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ABSTRACT

This volume is a teacher's edition in a series of books that contain open-ended exploration activities and experiments. These activities allow and encourage students to set their own goals, use their own creativity and ideas, investigate the wonders of nature, learn about the workings of real businesses, and draw conclusions from their investigations of these real-life situations. Students participate in many explorations by first making things such as a conic section through paper folding or a model of the path of a projectile and then using their creations to complete the exploration or investigation. The 40 activities are grouped under the National Council of Teachers of Mathematics' (NCTM) Curriculum Standards of: communications, spatial sense, measurement, number sense, connections, problem solving, and reasoning. Some activities include: The Normal Curve: Developing Normal Curves through Population Sampling; Models for Locus Theorems: Animating Locus Theorems; von Koch Snowflake: Understanding Sequences and Series, Football Arithmetic with Integers: Adding and Subtracting Positive and Negative Integers; The Mathematics of Medicine: Using Exponential Functions; VCR Counters: Modeling Reality with Quadratic Function; Am I Speeding? Discovering Linear Functions and Slope. Each activity contains a teacher's guide that lists: goal, student objectives, guide to the investigations, vocabulary, suggested path for remediation, and additional resources. Also included are: an introduction, purpose, materials needed, procedures, observations, conclusions, and suggestions for further study. (MKR)

GREAT EXPLORATIONS IN MATHEMATICS

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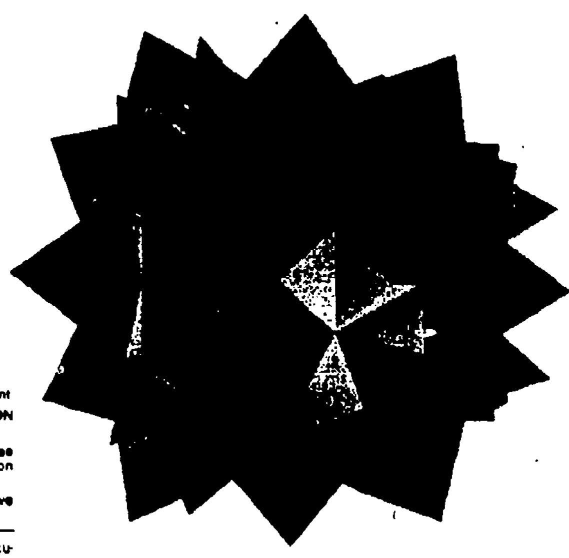

GRADES 9-12

TEACHER'S EDITION

Great Explorations in Mathematics

Grades 9-12
Teacher's Edition

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MEASUREMENT

NUMBER SENSE

CONNECTIONS

PROBLEM SOLVING

REASONING

Introduction to Great Explorations in Mathematics

A quote from *Everybody Counts* (National Research Council, 1989) effectively summarizes the philosophy of this book. "The focus of the secondary school curriculum remains—as it should—on the transition from concrete to conceptual mathematics." The investigations in this book begin at a concrete level where students are encouraged to experience mathematics and generalize from their experience. The activities we have provided can best be described as guided discovery lessons which allow students opportunities to learn mathematics experientially.

The National Council of Teachers of Mathematics advocated this approach in *Curriculum and Evaluation Standards for School Mathematics and Professional Standards for Teaching Mathematics.* The investigations we have provided here encourage students to develop their problem-solving skills, their ability to think logically, their ability to communicate mathematically, and their knowledge of mathematical connections among the various branches of mathematics and other disciplines, the sciences included.

These investigations do not fit the traditional definition of mathematics "problems." They tend to be open-ended. Very often the questions posed do not lend themselves to single, unique, "correct" answers. The answers may well depend on the data which the students collect, and these may vary widely with the conditions under which the data are collected. Our activities tend to follow a performance-based assessment model. This sort of model is strongly advocated by the National Council of Teachers of Mathematics and the Mathematical Sciences Education Board.

Although every investigation in the book was designed so that it may be done with a minimum of technology, many of them can be enhanced and/or carried out more easily with appropriate computer or calculator technology. We plan a further volume in this series which will address the use of just such technology in ways which will not do away with the student's need to know and understand the underlying mathematics but will hopefully enhance it. While we believe in the fullest use of available technology, we do not believe that it should or must inevitably get in the way of the students' acquisition of basic and advanced computational skills.

As the title suggests, *Great Explorations in Mathematics* is designed to be a journey, an exploration, and it is our hope that students and teachers will enjoy this journey as much as we have. Be prepared for some of these activities to raise more questions than they answer. That is our goal. In today's rapidly changing world, we believe that it is more important for students to pose questions and learn independently than it is for them to learn isolated mathematical facts, formulas, and procedures.

Bon Voyage!

About the Authors

Richard W. Dyches is Director of Test Development for National Computer Systems in Minneapolis, Minnesota where he is involved nationally with authentic assessment. A former elementary teacher and college professor, Dr. Dyches is actively involved in the National Council of Teachers of Mathematics and is a frequent speaker at workshops and conferences. Dr. Dyches attended the University of Alabama at Birmingham and Rutgers University and holds a doctoral degree in education.

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Wayland Y. Brown is a priest of the Catholic Diocese of Savannah and an instructor of Mathematics and Computer Science at Georgia Southern University in Statesboro, Georgia. He teaches calculator-driven College Algebra and Trigonometry and Calculus using Macintosh computers with Mathematica. A graduate of the University of Washington, Father Brown is active in planning in-service workshops for teachers.

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COMMUNICATIONS

TEACHER'S GUIDE
THE NORMAL CURVE

GOAL: To develop students' understanding of the concept of normally distributed data.

STUDENT OBJECTIVES:

- ✓ To conduct an experiment to produce data which will graph as a normal curve.
- ✓ To explore properties of the normal curve.

GUIDE TO THE INVESTIGATIONS: The prerequisites for this investigation include some knowledge of measures of central tendency (mean, median, mode), the ability to compute percents, and the ability to graph data points on a rectangular coordinate system. This activity is ideal for students working in cooperative groups. They will need adequate supplies and sufficient work space. Graphs produced by the various cooperative groups may be displayed in the classroom or other appropriate locations within the school.

VOCABULARY: x–axis, y–axis, data, graph, bell–shaped curve, normal distribution, normal curve

SUGGESTED PATH FOR REMEDIATION: Due to the straight–forward nature of this activity it is not usual for students to require remediation. It is, however, possible that some students may require additional work with constructing graphs and plotting points.

ADDITIONAL RESOURCES: This activity is described in *A Handbook of Aids for Teaching Junior–Senior High School Mathematics* by Stephen Krulik (W. B. Saunders Company, 1971), which is currently out of print. An excellent source of statistical instructional materials is the *Quantitative Literacy Series: Exploring Data* by James M. Landwehr and Ann E. Watkins (Dale Seymour Publications).

THE NORMAL CURVE

INTRODUCTION: Researchers often find that data are distributed in such a way as to produce a normal or bell-shaped curve.

PURPOSES:

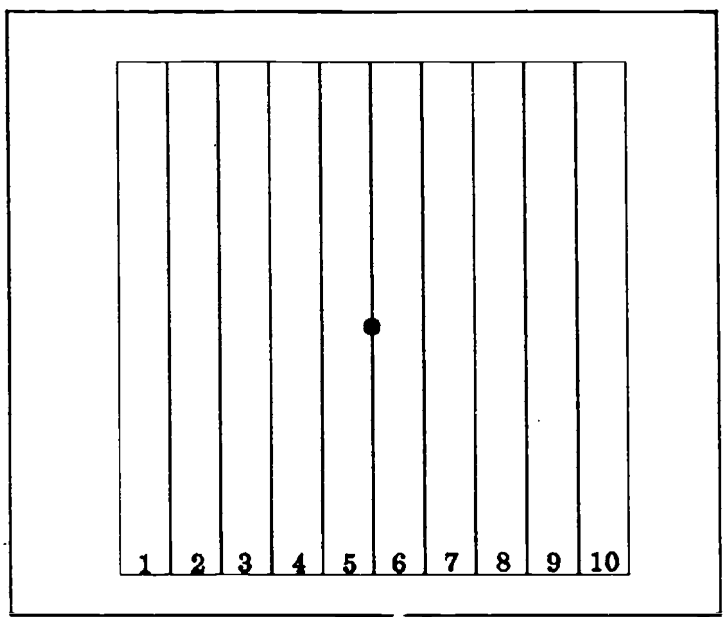
- ✓ Can you create a situation in the classroom which will model a normal distribution?
- ✓ What are some of the properties of this distribution?

MATERIALS:

a ten-inch square of butcher paper or poster board
ruler
uncooked long-grain rice
graph paper (enough to make 11 graphs)

PROCEDURES:

1. Use a ruler to cut a ten-inch square from butcher paper or poster board, and mark this square into one inch columns. Number the columns from left to right one through ten. Place a large dot in the center of the paper.
2. Count out a hundred grains of uncooked long-grain rice. Lay the ruled paper on the floor and stand over it. Drop the rice a few grains at a time over the dot in the center of the paper.



1 - 1 *A ten-inch square with one inch columns*

3. After all the rice has been dropped, count the number of grains of rice that landed in each of the ten columns. Record your count on the following table. If a grain is on a line, count it as belonging to the column containing more grains of rice.

Column	# of grains
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

4. Construct a graph with the horizontal axis representing the column numbers from one through ten and the vertical axis representing the number of grains in each column. Plot the ten points generated in your experiment, and connect them with as smooth a curve as possible.
5. Repeat this experiment ten times, varying the height from which you drop the rice. Plot a graph for each experiment.
6. Combine the totals for each column from all ten experiments and record them on the table below. Produce a graph from these totals.

Total of Experiments

Column	# of grains
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

OBSERVATIONS:

1. Compute the percent of grains in each of the ten columns from the data for the ten experiments combined, and complete the table below.

Column	# of grains	% of rice
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Total		100%

2. What percent of the grains is in one of columns 2–9 combined?

   Answers will vary. More than 90%.

3. What percent of the grains is in one of columns 3–8 combined?

   Answers will vary. Between 66⅔% and 90%.

4. What percent of the grains is in one of columns 4–7 combined?

   Answers will vary. ≈ 66⅔%

5. What percent of the grains is in one of columns 5–6 combined?

   Answers will vary. Almost 50%.

6. What is the probability that a grain of rice will land in column 5 or 6?

   Answers will vary. ≈ 50%

7. What is the probability that a grain of rice will land in column 1 or 10?

   Answers will vary. < 10%

8. What is the probability that a grain of rice will land in one of columns 1–5?

   Answers will vary. ≈ 50%

9. What is the probability that a grain of rice will land in one of columns 6–10?

   Answers will vary. ≈ 50%

CONCLUSIONS:

1. List some of the characteristics of a normal distribution.

There are more responses in the central area than on either end.

The graph should be bell-shaped. It extends indefinitely in both directions.

2. Give some examples of sets of normally distributed data.

The heights of a large population of people.

The weights of a large population of people.

The IQ's of a large population of people.

SUGGESTIONS FOR FURTHER STUDY:

- Survey a hundred persons in the same age group, recording their height. Use the data collected in this survey to construct a graph. Would you say that these data are normally distributed? Why or why not?

TEACHER'S GUIDE
THE GOLDEN RATIO

GOAL: The student will develop an understanding of the historical significance of the Golden Ratio.

STUDENT OBJECTIVES:

- ✓ To construct an example of a Golden Rectangle.
- ✓ To construct an example of a rectangle different from a Golden Rectangle.
- ✓ To survey a hundred people to determine which of these rectangles they find most visually appealing.

GUIDE TO THE INVESTIGATIONS: Knowledge of how to perform basic geometric compass and straight edge constructs and an understanding of ratio and proportion are prerequisite for this activity.

One way to manage this investigation is to have each student construct an example of and a counter-example to a Golden Rectangle, then have the class work together to survey a large number of people to see which rectangle they find more pleasing to the eye. If the class has 25 students, each class member could ask four people, for a sample size of a hundred.

VOCABULARY: Golden Ratio, Golden Rectangle, ratio, compass, straight edge, construct, square, vertices, line segment, bisect, midpoint, radius, arc, intersects, ray, perpendicular

SUGGESTED PATH FOR REMEDIATION: It may be necessary to review basic geometric constructions including bisecting a line segment and constructing a perpendicular to a line at a point on the line.

ADDITIONAL RESOURCES: *Historical Topics for the Mathematics Classroom* from the National Council of Teachers of Mathematics is an excellent resource for activities that relate to the history of mathematics.

The Golden Ratio!

INTRODUCTION: The Golden Ratio is $\frac{\sqrt{5}+1}{2}$. A Golden Rectangle is a rectangle in which the ratio of the length to the width is the Golden Ratio. Historically Golden Rectangles have been said to be more pleasing to the eye than other rectangles.

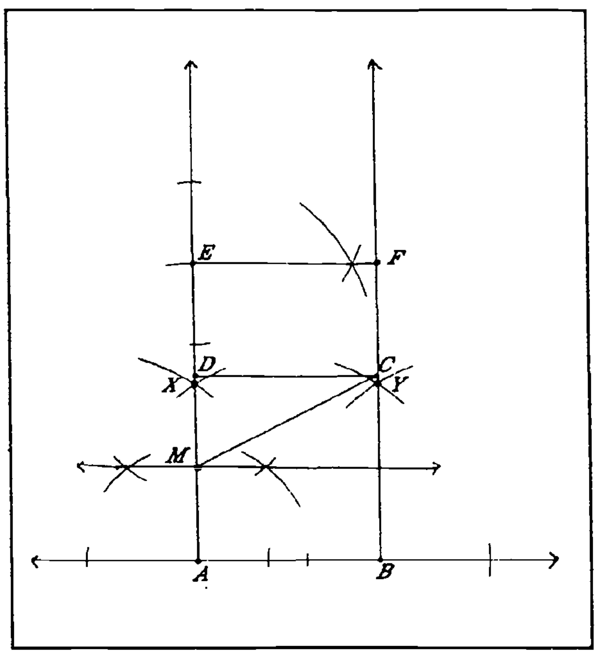
PURPOSES:

- ✓ How can you construct a Golden Rectangle?
- ✓ Is a Golden Rectangle really more pleasing to the eye that other rectangles?

MATERIALS:

straight edge
compass
unlined paper

PROCEDURES:

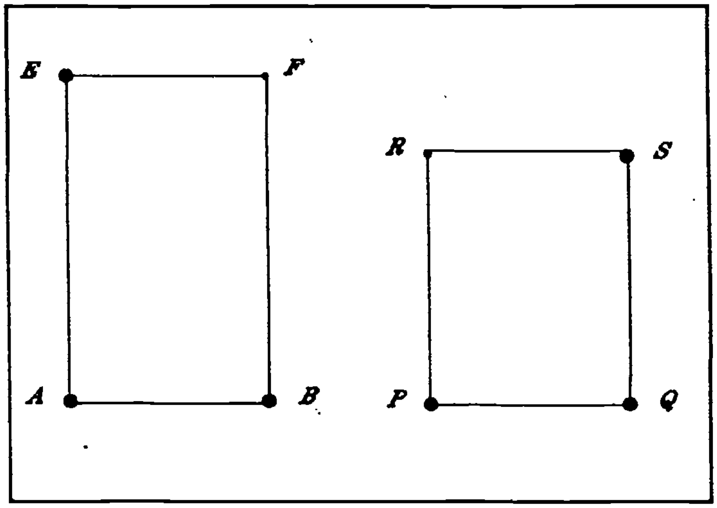


2 - 1 *Rectangle ABFE is a Golden Rectangle.*

1. To construct a Golden Rectangle, start by constructing a square. Draw a line, and label two points on the line A and B. Construct a perpendicular to line AB at point A, and label another point on this perpendicular line X. Construct a perpendicular to line AB at point B. Label another point on this perpendicular line Y. Construct a copy of line segment AB on line AX, using point A as one of the endpoints of the copy. Label the other endpoint D. Construct a copy of line segment AB on line BY, using point B as one of the endpoints of the copy. Label the other endpoint C. Draw line CD. ABCD is a square. Bisect line segment AD and label the midpoint M. Draw line segment MC. Place the compass on point M and set the radius at the length of line segment MC. Draw an arc that intersects ray MD. Label the point of intersection E. Construct a line segment perpendicular to line AD at point E. This perpendicular line intersects ray BC. Label the point of intersection F. The rectangle ABFE is a Golden Rectangle.

2. Trace the Golden Rectangle that you constructed on a sheet of paper so that the construction marks are not present. On this same sheet of paper draw a rectangle in which the ratio of length to width is *not* the Golden Ratio.

3. Survey a hundred or more persons to see which rectangle they prefer. Tally responses in the table below.



2-2

Prefer *ABFE*	Prefer *PQRS*

OBSERVATIONS:

1. Use the ruler to determine the measure of line segment AB.

   Answers will vary.

2. Use the ruler to determine the measure of line segment MC.

   Answers will vary dependent on the length of AB. $MC = \frac{\sqrt{5}}{2}(AB)$

3. Write a decimal approximation of the ratio of the measure of line segment MC to the measure of line segment AB.

   ≈ 1.118

4. What is the decimal approximation of $\frac{\sqrt{5}}{2}$?

   ≈ 1.118

5. Use a ruler to determine the measure of line segment *AE*.

Answers will vary. $\left(\frac{\sqrt{5}+1}{2}\right)(AB)$

6. Write a decimal approximation of the ratio of the measure of line segment *AE* to the measure of line segment *AB*.

≈ 1.618

7. What is the decimal approximation of the Golden Ratio$\left(\frac{\sqrt{5}+1}{2}\right)$?

≈ 1.618

CONCLUSIONS:

1. Prove that rectangle *ABFE* is a Golden Rectangle.

The ratio of AE to AB (length to width) is $\approx \left(\frac{\sqrt{5}+1}{2}\right)$.

2. Did the results of the survey support the contention that a Golden Rectangle is more pleasing to the eye than other rectangles?

Answers will vary.

3. What might be some of the consequences if a Golden Rectangle is really more eye catching than other types of rectangles?

Make packages with this ratio, so more people will choose the product.

SUGGESTIONS FOR FURTHER STUDY:

- Measure the length and width of a variety of commercial packages. Are manufacturers using the Golden Ratio in designing their packaging?
- Do a similar project using the Golden Triangle. A Golden Triangle is an isosceles triangle in which the ratio of the length of the legs to the length of the base is the Golden Ratio. How will you construct a Golden Triangle?

TEACHER'S GUIDE
THE BIRTHDAY PROBLEM

GOAL: The student will develop an understanding of experimental and theoretical probability.

STUDENT OBJECTIVES:

- ✓ To find the experimental probability that at least two people in a group of size n have the same birthday, where n is a natural number greater than or equal to two.
- ✓ To find the theoretical probability that at least two people in a group of size n have the same birthday.
- ✓ To compare the experimental and theoretical probabilities for this event.
- ✓ To answer the question, "How many people would you have to have in a group before you could be *relatively certain* that at least two people would share the same birthday"?

GUIDE TO THE INVESTIGATIONS: Some knowledge of probability is a prerequisite for this activity. It can be done as an individual project, a cooperative group project, or as a class project. The investigation asks students to conduct surveys, but if it is impractical, an alternative is to use reference books of the "Who's Who in..." type. This sort of book usually gives dates of birth. Students may randomly select people from the entries of such a book and record the birthdays of those whom they select.

VOCABULARY: experimental (empirical) probability, theoretical probability, sample, natural number

SUGGESTED PATH FOR REMEDIATION: Students may have difficulty with theoretical probability. One possible way to make this task easier is to ask a different question. Change, "What is the theoretical probability that in a group of fifteen people at least two people will have the same birthday?" to "What is the theoretical probability that in a group of fifteen people no two people have the same birthday?" The latter question is easier to answer, and its answer may be subtracted from one to answer the former.

ADDITIONAL RESOURCES: An excellent source of investigations related to probability is the *Quantitative Literacy Series, The Art and Techniques of Simulation* (Dale Seymour Publications).

THE BIRTHDAY PROBLEM

INTRODUCTION: Omitting February 29, there are 365 possible birthdays. Given any two people, the probability that they have the same birthday is $\frac{1}{365}$. What is the probability that at least two persons in a group of n have the same birthday, where n is a natural number greater than two?

PURPOSES:

✓ Can you develop a technique that will allow you to find the experimental probability that at least two persons in a group of n have the same birthday, where n is a natural number greater than two?

✓ Can you compute the theoretical probability that at least two persons in a group of n have the same birthday, where n is a natural number greater than two?

✓ How do the two compare?

✓ Clearly, when you increase the number n of people in the group, it becomes more probable that at least two of them will have the same birthday. How large must n be before you can be reasonably certain that at least two people will, in fact, have the same birthday?

MATERIALS:

calculator
one of the following:
people to poll for birth dates
reference sources such as "Who's Who in..." books

PROCEDURES:

1. What do you guess will be the probability that at least two persons in a group of fifteen will have the same birthday? Twenty-five? Sixty? Record your guesses below before you begin the activity.

n	Guess
15	*Answers will vary.*
25	
60	

2. Survey eight different groups of fifteen people each to record their birthdays. You might ask fifteen people in the hall before school, in the cafeteria at lunch, or on the school bus. Ask only the month and day. Record your findings on the following table.

Group #	Number with the same birthday
1	*Answers will vary.*
2	
3	
4	
5	
6	
7	
8	

3. Calculate the theoretical probability that at least two persons in a group of fifteen will have the same birth date. If you do not know how to do this, review probability in a mathematics text.

4. Repeat steps 2 and 3 using a group of twenty-five people. If you do not want to survey actual people, your teacher will be able to suggest alternative methods for you to obtain a sample. Repeat this sampling procedure eight times, recording your findings in the chart below.

Group #	Number with the same birthday
1	*Answers will vary.*
2	
3	
4	
5	
6	
7	
8	

5. Repeat steps 2 and 3 using eight groups of sixty people, recording the findings below.

Group #	Number with the same birthday
1	*Answers will vary.*
2	
3	
4	
5	
6	
7	
8	

OBSERVATIONS:

1. What did you find to be the empirical probability that at least two of fifteen people have the same birthday?

   Answers will vary. Nearly .25 ($\frac{2}{8}$) might be expected.

2. What did you find to be the theoretical probability that at least two of fifteen people have the same birthday?

   1 – P(no two do) = 1 – 0.747 = 0.253

3. What did you find to be the empirical probability that at least two of twenty-five people have the same birthday?

   Answers will vary, nearly .5 ($\frac{4}{8}$) might be expected.

4. What did you find to be the theoretical probability that at least two of twenty-five people have the same birthday?

   1 – P(no two do) = 1 – 0.431 = 0.569

5. What did you find to be the empirical probability that at least two of sixty people have the same birthday?

   Answers will vary. Nearly .95 ($\frac{7}{8}$ or $\frac{8}{8}$) might be expected.

6. What did you find to be the theoretical probability that at least two of sixty people have the same birthday?

   $1 - \frac{365!}{305!\,(365)^{60}} = 1 - 0.006 = .994$

CONCLUSIONS:

1. How do the empirical results, theoretical numbers, and your own guesses compare one with another for each of the three group sizes?

   Answers will vary. The more trials that are made the closer they should be.

2. For what *n* will the theoretical probability be 0.50?

   For n = 23, probability is .507. For n = 22, probability is .476.

3. For what *n* will the theoretical probability be 1.00?

   It will never be exactly one.

4. For what *n* will the theoretical probability be 0.99?

   For n = 56, probability is .989. For n = 57, probability is .9901

SUGGESTIONS FOR FURTHER STUDY:

- If your state has a lottery, investigate the rules of the lottery and calculate the probability of winning. Compare your answer with the published chances of success.
- Call the meteorologist of a local television station and invite him/her to visit your school and explain how rain is forecast. What does it mean when the forecast is a thirty-percent chance of rain?

TEACHER'S GUIDE
GETTING AROUND TOWN

GOAL: The student will develop an understanding of the concept of a traversable network and discover the relationship among the numbers of vertices, regions, and arcs of networks.

STUDENT OBJECTIVES:

- ✓ To discover the conditions under which a network is traversable.
- ✓ To discover relationships among the number of vertices, regions, and arcs of networks.

GUIDE TO THE INVESTIGATIONS: The prerequisites for this activity are an understanding of the terms path, vertex, region, and arc.

Students must learn to find mathematical information in libraries. This investigation requires them to research the work of the Swiss mathematician Leonhard Euler. There are a large number of published references to his work with networks. Students may work independently or in groups, and they should be encouraged to communicate their results both orally and in writing.

VOCABULARY: network, vertex, vertices, path, arc, region, traversable, inductive reasoning, deductive reasoning

SUGGESTED PATH FOR REMEDIATION: If students do not see the relationship among the number of vertices, regions, and arcs, check that they have correctly counted and recorded the information on the examples they have produced. This can be done by having students check each others' data. Check to be sure that, when they are counting the number of regions, students count the area outside the network as one.

ADDITIONAL RESOURCES: *A History of Mathematics: An Introduction*, by Victor J. Katz (Harper Collins, 1993) and *For all Practical Purposes: An Introduction to Contemporary Mathematics*, by Lynn A. Steen (W. H. Freeman and Company, 1988).

GETTING AROUND TOWN

INTRODUCTION: A network in a plane consists of points, called **vertices**, paths connecting the vertices, called **arcs**, and regions bounded by the arcs. That part of the plane exterior to the arcs is also a **region**. A network is said to be traversable if it can be traced without lifting the pencil from the paper or passing over any path more than once.

PURPOSES:

✓ Under what conditions will a network be traversable?

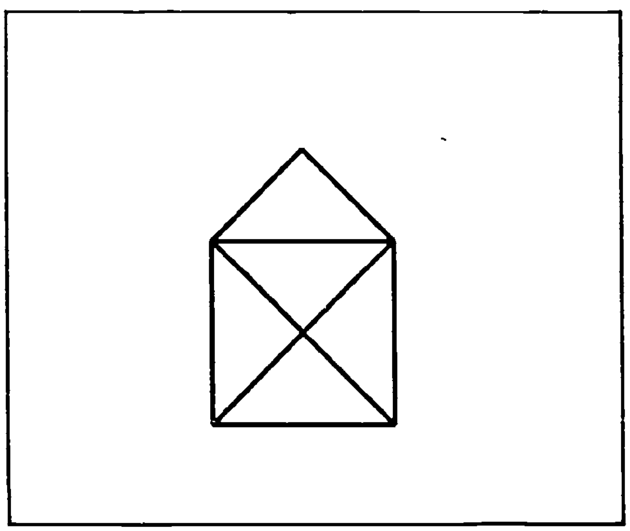
✓ What are the relationships among the numbers of vertices, regions, and arcs of traversable and nontraversable networks?

MATERIALS:

Euler reference works
poster board
markers
straight edge

PROCEDURES:

1. Can you draw the figure in illustration 4-1 without picking up your pencil or retracing any line segment?
2. Research the work of Euler, and prepare a report on his findings related to networks, especially the conditions under which networks are traversable.
3. Prepare a poster showing several examples of traversable networks.
4. Prepare a poster showing several examples of nontraversable networks.



4 - 1 *Is this traversable?*

5. Continue your exploration by completing the activities in the Observation section.

OBSERVATIONS:

1. Count the number of vertices, regions, and arcs in each of your examples of traversable networks. Record these data in the table below.

Traversable Network Data

Example	Vertices	Arcs	Regions
1	*Answers will*		
2	*vary.*		
3			
4			
5			
6			
7			
8			
9			
10			

2. Repeat for nontraversable networks and record these data below.

Nontraversable Network Data

Example	Vertices	Arcs	Regions
1	*Answers will*		
2	*vary.*		
3			
4			
5			
6			
7			
8			
9			
10			

CONCLUSIONS:

1. Write a formula that expresses the relationship among the number of vertices, regions, and arcs.

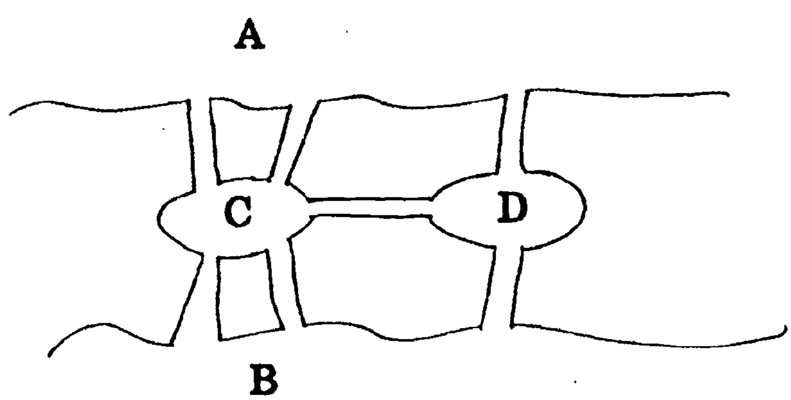
Answers will vary. $V + R - 2 = A$

2. Your investigation uses inductive reasoning to form a hypothesis about this relationship. Try to show deductively that your formula is always true.

   Answers will vary.

SUGGESTIONS FOR FURTHER STUDY:

- The Koenigsberg Bridge problem is the famous problem which Euler solved. After Sunday Mass the citizens of Koenigsberg were in the habit of strolling over the seven bridges that connected the mainland to two islands in the river which flowed through town. They amused themselves by trying to find somewhere in town from which they could set out and walk over every bridge without doing so twice. Were they able to accomplish this task? Why, or why not?



SPATIAL SENSE

TEACHER'S GUIDE PANTOGRAPH

GOAL: To develop students' understanding of similarity.

STUDENT OBJECTIVES:

✓ To construct an instrument called a pantograph to produce a geometric figure similar to a given figure.

✓ To use the pantograph to construct similar polygons and investigate the properties of similarity.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity include an ability to identify corresponding parts of similar figures, measure line segments, measure angles, and understand the terms polygon, proportional, and congruent.

Although each student may want to construct individual pantographs, they should be encouraged to work in pairs to use the pantograph to construct the similar figure. Two pairs of hands are almost necessary to use the pantograph. A large flat surface is essential.

VOCABULARY: ratio, polygon, line segment, angle, proportion, congruent, corresponding angles, corresponding sides, similarity

SUGGESTED PATH FOR REMEDIATION: If students have trouble setting up the appropriate correspondence relationships, they can be encouraged to use a system for labeling the vertices of figures to assist in maintaining the correct correspondence relationship. For example, if the original figure has vertices labeled X, Y, and Z, then, as the pantograph is used to construct the similar figure, label it in the following manner. When point B of the pantograph is on vertex X, label the location of point C on the pantograph as X'. Similarly, when point B of the pantograph is on vertex Y, label the location of point C on the pantograph as Y'. When point B of the pantograph is on vertex Z of the figure, label the location of point C of the pantograph as Z'. Angle X will correspond to angle X', line segment XY will correspond to line segment $X'Y'$, and so forth.

ADDITIONAL RESOURCES: Investigations about similarity can be significantly aided by computer software, such as Logo, the Geometric Supposer (Sunburst), and the Geometric Sketch Pad (IBM).

PANTOGRAPH

INTRODUCTION: Two geometric figures are similar, intuitively speaking, if they have the same shape. They need not, however, have the same size. Formally, two polygons are similar if there exist correspondence relations between the two of them such that all corresponding angles are congruent and all corresponding sides are proportional.

PURPOSES:

✓ How can you construct a device that will allow you to produce a figure similar to any given figure?

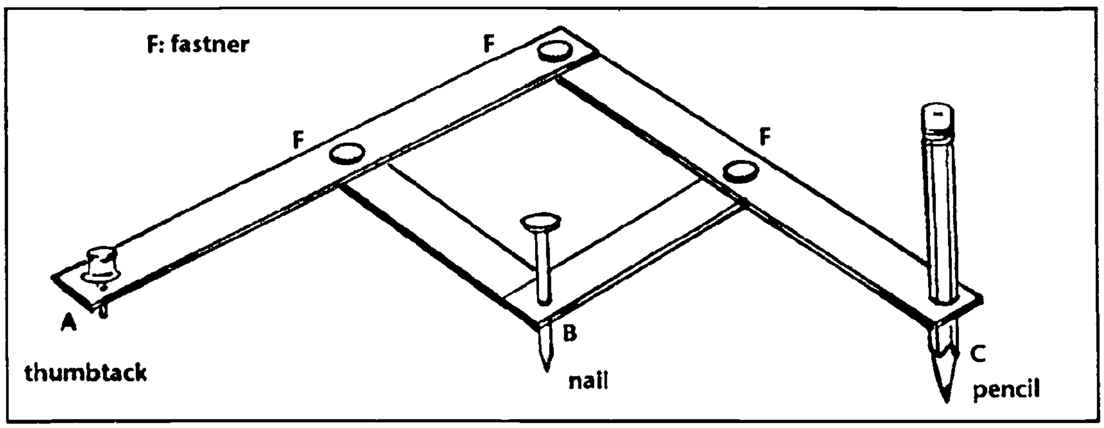
✓ What are some characteristics of similar polygons?

MATERIALS:

oak tag, heavy poster board, or cardboard
scissors
brass fasteners
thumbtack
small nail
pencil
butcher paper
ruler
protractor

PROCEDURES:

1. From the oak tag, heavy poster board, or cardboard, cut four strips, each one inch wide. Two of the strips should be twelve inches long and the other two about six inches long. Join the four pieces with the brass fasteners as shown in the illustration 5-1 below.



5 - 1 *Constructing a pantograph*

If the short pieces are attached to the long pieces in the midpoint of the long pieces then the ratio of similarity will be 2:1. Thus, changing the position at which the short pieces are attached to the long pieces will alter the ratio of similarity.

Through the point marked *A* on the diagram insert the thumb tack, which will be used to keep this point fixed when constructing similar figures. Through the point marked *B* in the diagram insert the small nail, which will be used to trace around the figure for which a similar figure is to be constructed. Through the point marked *C* in the diagram insert the pencil, for this is the arm of the device which will do the drawing.

2. On a large piece of butcher paper draw a polygon for which you wish to produce a similar figure. Use the thumb tack to fix point *A*. Place point *C* on the area of the paper where the similar figure will be drawn. With the point of the nail at point *B*, trace around the given figure. It will be easier to do if you work with a partner. One of you will trace the original polygon with the nail, while the other handles the pencil, making sure it stays in contact with the paper.

OBSERVATIONS:

1. Measure the angles in the original figure and the angles in the similar reproduction. Record these measurements in the table below.

Angle	Measure	Corresponding Angle	Measure

2. Are the corresponding angles congruent?

They are.

3. Measure the lengths of the line segments in the original figure and the similar reproduction. Record these measures in the table below.

Line Segment	Measure	Corresponding Segment	Measure

4. Are the lengths of the corresponding sides proportional?

Yes.

5. What is the ratio of the length of the sides of the original figure to the length of the corresponding sides of the similar figure?

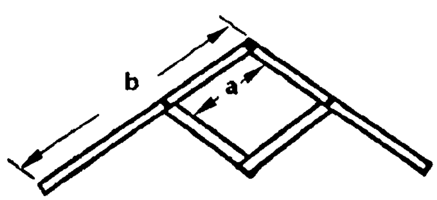
Answers will vary.

CONCLUSIONS:

1. How does the ratio of similarity relate to the construction of the pantograph?

The locus at which the fasteners are inserted will determine the ratio.

2. Make a drawing to show how you could construct a pantograph that will produce similar figures with the ratio of similarity of 2:3. Label your drawing.



a/b = 1/3

33

SUGGESTIONS FOR FURTHER STUDY:

- Investigate similar figures using an overhead projector. Draw a polygon on a transparent sheet. Project this image onto a piece of butcher paper taped to the wall. Trace this image onto the butcher paper. Compare the measures of the original figure with those of the similar reproduction. What is the ratio of similarity?
- Measure the distance from the overhead projector to the paper, and find the ratio of similarity. Move the projector and repeat this process. How does the placement of the overhead projector in relationship to the wall change the ratio of similarity?

TEACHER'S GUIDE
MODELS FOR LOCUS THEOREMS

GOAL: To develop students' understanding of locus theorems from geometry.

STUDENT OBJECTIVES:

- ✓ To construct a model for the theorem: The locus of points equidistant from two parallel lines is a third line parallel to the two given lines and midway between them.
- ✓ To construct a model for the theorem: The locus of points equidistant from the endpoints of a line segment is the perpendicular bisector of the line segment.
- ✓ To construct a model for the theorem: The locus of points equidistant from a given point is a circle.
- ✓ To construct a model for the theorem: The locus of the vertex of a right triangle with a fixed hypotenuse is a semicircle with the hypotenuse as diameter.
- ✓ To construct a model for the theorem: The locus of points equidistant from the sides of an angle is a line which bisects the angle.

GUIDE TO THE INVESTIGATIONS: A working knowledge of the vocabulary of geometry is a prerequisite for this investigation. Students may work individually or in cooperative groups. If a video camera is available, each card may be taped briefly to illustrate the animation effect.

VOCABULARY: locus, points, line, line segment, angle, equidistant, parallel, perpendicular, bisector, circle, vertex, right triangle, hypotenuse, semicircle

SUGGESTED PATH FOR REMEDIATION: Some students may have difficulty interpreting the vocabulary of geometry and understanding locus theorems. It may be necessary to provide additional explanation or examples for some of the theorems.

ADDITIONAL RESOURCES: Most, if not all, high school and geometry texts will have helpful materials to assist your preparation for this exercise.

MODELS FOR LOCUS THEOREMS

INTRODUCTION: You can think of locus theorems in geometry as describing paths of moving points which leave traces as they move.

PURPOSE

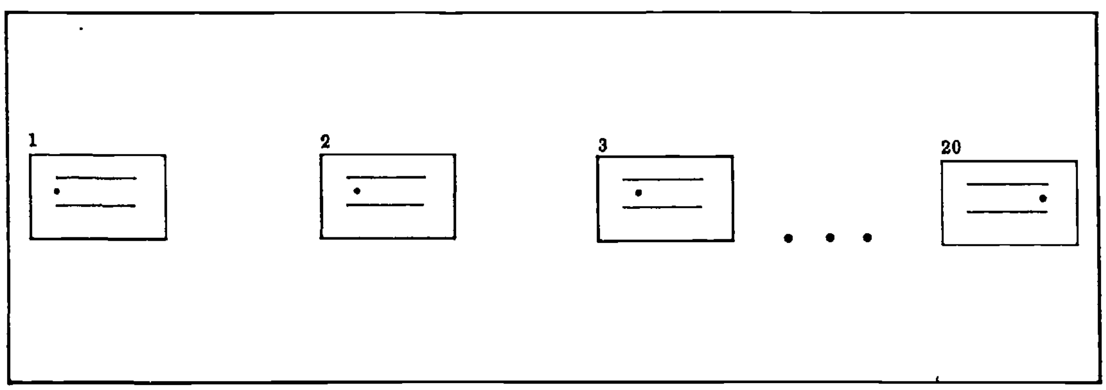
✓ Can animation techniques be used to model locus theorems?

MATERIALS:

three-by-five index cards (100 per student or group)
paper fasteners
ruler
protractor
compass
video camera and VCR (optional)

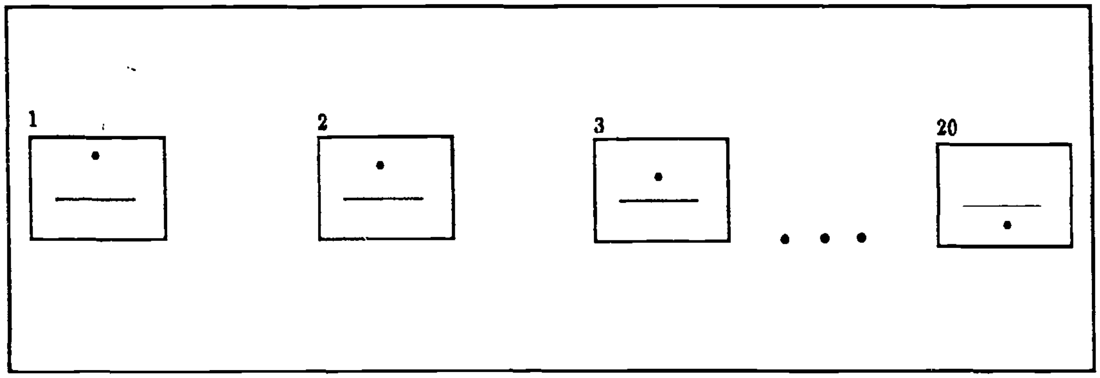
PROCEDURES:

1. Construct a model for the theorem: The locus of points equidistant from two parallel lines is a third line parallel to the two given lines and midway between them. For this model use twenty index cards. On each card draw identical sets of parallel segments. On the first card place a point equidistant from the left endpoints of the two line segments. On each of the consecutive cards place a point equidistant from the two parallel lines, moving the point more to the right on successive cards. See illustration 6-1. Use the paper fastener to clip the cards together in order in the upper left hand corner.



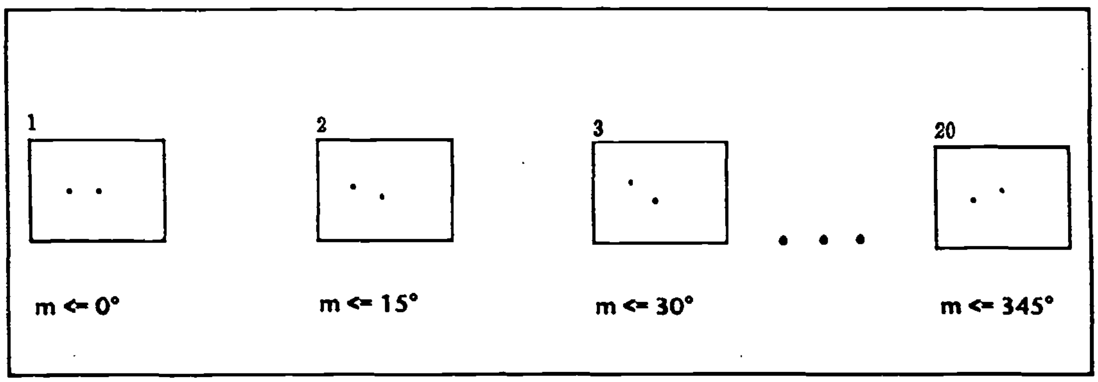
6 - 1

2. Construct a model for the theorem: The locus of points equidistant from the endpoints of a line segment is the perpendicular bisector of the line segment. See illustration 6–2.



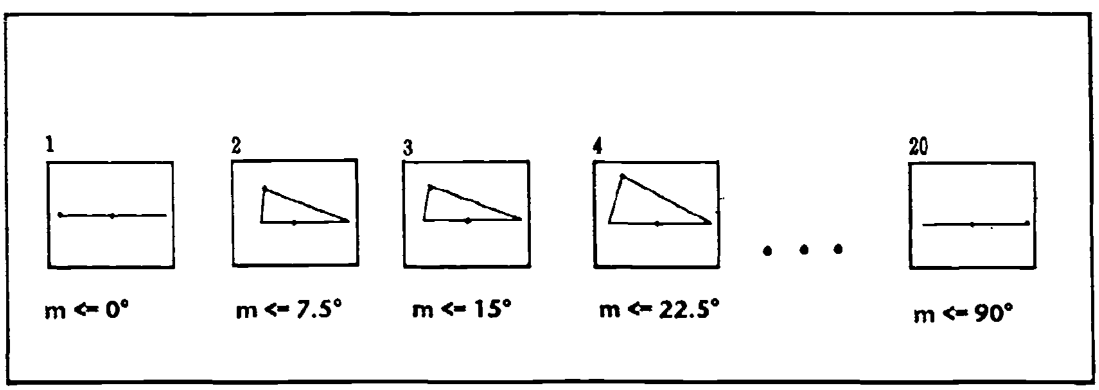
6 - 2

3. Construct a model for the theorem: The locus of points equidistant from a given point is a circle. See illustration 6–3.



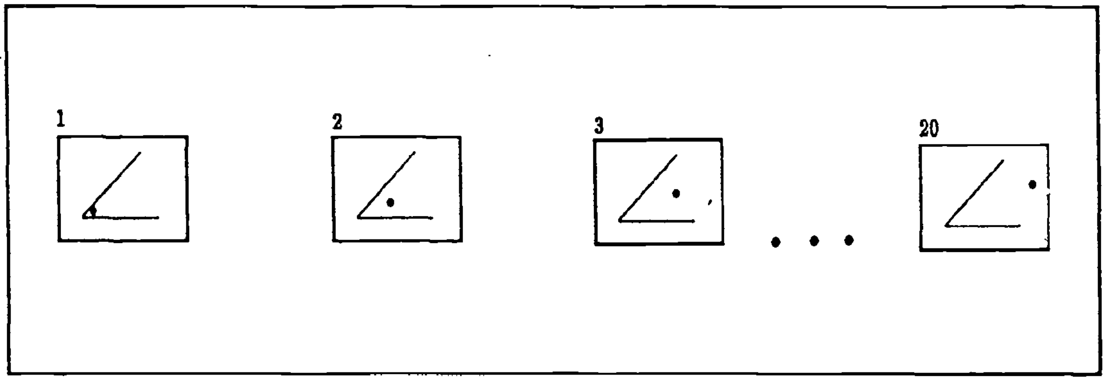
6 - 3

4. Construct a model for the theorem: The locus of the vertex of a right triangle with a fixed hypotenuse is a semicircle with the hypotenuse as the diameter. See illustration 6–4.



6 - 4

5. Construct a model for the theorem: The locus of points equidistant from the sides of an angle is a line which bisects the angle. See illustration 6-5.



6-5

6. Hold each set of cards in the corner where they are clipped together, and flip through the cards as rapidly as possible, trying to avoid skipping any of them.

OBSERVATIONS:

1. Describe what you perceive as you flip through the set of cards for the theorem: The locus of points equidistant from two parallel lines is a third line parallel to the two given lines, and midway between them.

   The point appears to be moving along in a parallel line between the two lines.

2. Describe what you perceive as you flip through the set of cards for the theorem: The locus of points equidistant from the ends of a line segment is the perpendicular bisector of the line segment.

   The point appears to be moving from the top of the card to the bottom.

3. Describe what you perceive as you flip through the set of cards for the theorem: The locus of points equidistant from a given point is a circle.

   The point appears to be moving in a circle around the marked center.

4. Describe what you perceive as you flip through the set of cards for the theorem: The locus of the vertex of a right triangle with a fixed hypotenuse is a semicircle with the hypotenuse as diameter.

   The triangle appears to be moving in a semicircle.

5. Describe what you perceive as you flip through the set of cards for the theorem: The locus of points equidistant from the sides of an angle is the line which bisects the angle.

   The point appears to be moving from the vertex of an angle along its bisector.

CONCLUSIONS:

1. Describe how using a compass to construct a circle is related to the theorem: The locus of points equidistant from a given point is a circle.

   The pencil point is always at a fixed distance from the compass point.

2. Describe how the construction of the perpendicular bisector is related to the theorem: The locus of points equidistant from the ends of a line segment is a perpendicular bisector of the line segment.

   Two points are found, one above and one below the segment that are equidistant from the end points of the segment. When these two points are connected, all the points on the line that is created are equidistant from the end points.

3. Describe how the construction of the bisector of an angle is related to the theorem: The locus of points equidistant from the sides of an angle is the bisector of the angles.

   Points are found on each ray that are equidistant from the vertex, then a point is found that is equidistant from the two points. When this third point is connected to the vertex, the points are all equidistant from the sides of the angle.

SUGGESTIONS FOR FURTHER STUDY:

- Write a locus theorem related to the construction of a line perpendicular to a given line through a point not on the line.
- Identify other locus theorems and construct models for them.

TEACHER'S GUIDE
PARABOLAS THROUGH PAPER-FOLDING

GOAL: To develop students' understanding of the definition of the parabola and the significance of the directrix and focus.

STUDENT OBJECTIVE:

✓ To construct parabolas through paper-folding and examine how varying the placements of the directrix and focus change the resulting parabola.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity include knowledge of the definitions of tangent line, parabola, directrix, and focus. We recommend that each student be allowed to construct parabolas through paper-folding. When students begin to investigate the effects of placing the focus various distances from the directrix, we recommend that they work in groups, with each group member choosing a different placement of the focus in relation to the directrix.

VOCABULARY: parabola, directrix, focus, tangent line, equidistant, conic section

SUGGESTED PATH FOR REMEDIATION: If students have difficulty developing an intuitive feel for the definition of a parabola after completing the paper folding activity, you may want to try the following course of remediation. Take one of the parabolas constructed through paper folding, and select a point on the parabola. Using a ruler, measure the distance from this point to the focus of the given parabola and the perpendicular distance from this point to its directrix. These two distances should be equal. Have students select several distinct points on the parabola and make like measurements. They will then see that the sum of these two distances does not depend on the point selected, and they should then begin to appreciate the crux of the definition.

ADDITIONAL RESOURCES: Lines of symmetry for a parabola can be explored using a device called a Mira. Examples of using this device to locate lines of symmetry can be found in *Mira Math Activities for High School: A New Dimension in Motivation and Understanding*, a Mira Math Company publication.

PARABOLAS THROUGH PAPER-FOLDING

INTRODUCTION: A **parabola** is a U-shaped curve, defined as the set of all points equidistant from a fixed line, called the directrix, and a fixed point, called the focus. The conic section called parabola can be constructed from the definition by producing an envelope of lines tangent to the curve.

PURPOSES:

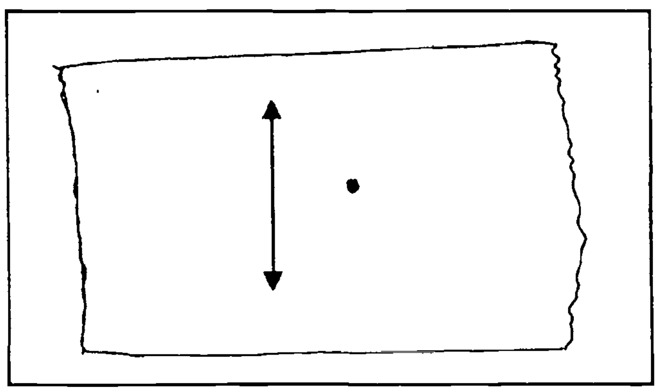
- ✓ Given a line and a point not on the line, how can paper folding be used to outline the parabola that has the given line as its directrix and the given point as its focus?
- ✓ How does varying the distance from the focus to the directrix change the resulting parabola?

MATERIALS:

wax paper
felt-tip marking pen
several sheets of 8.5" X 11" construction paper
tape

PROCEDURES:

1. Tear off a piece of wax paper about twelve to fifteen inches long. Use a straight edge to draw a line on the wax paper with the felt-tip pen. This line will be the directrix of the parabola you are about to construct. Choose a point not on the directrix, and mark it with the felt-tip pen. It will be the focus of the parabola. (See illustration 7-1.)



7-1

2. Fold the paper so that the focus lies over the directrix, and crease the paper carefully. This crease is a line tangent to the parabola determined by the directrix and focus that you have chosen. This means that one point on this line is on the parabola. Now move the focus to a different point over the directrix, and crease again. Continue this process until the focus has been moved all along the directrix. The more creases (tangent lines), the sharper the outline of the parabola will be.

3. To help bring out the outline of the parabola enveloped by the creases (tangent lines), mount the wax paper on a piece of construction paper. To do this, lay construction paper under the wax paper on which the paper folding was done. Fold the edges of the wax paper under the edges of the construction paper, and tape it securely in place.
4. Perform the paper folding activity several times, each time with different distances from the focus to the directrix. Note how the changes you make change the shape of the resulting parabola.

OBSERVATIONS:

1. What happens to the width of the parabola if the focus is moved closer to the directrix?

   The parabola becomes more narrow.

2. What happens to the width of the parabola if the focus is moved farther away from the directrix?

   The parabola becomes wider.

CONCLUSIONS:

1. Write a paragraph describing the relationship between the distance between the focus and the directrix to the width of the parabola they determine.

   The width of the parabola $y = ax^2$ depends on the coefficient of a. If the distance between the focus and directrix is d, then $a = \frac{1}{2d}$. Thus the width of the parabola varies inversely with the distance between the focus and the directrix.

SUGGESTIONS FOR FURTHER STUDY:

- On a rectangular coordinate system, consider the line $y = 5$ as the directrix and the point $(5, -2)$ as the focus. Use the definition of a parabola to write an equation for the parabola defined by the given directrix and focus.
- Use the definition of a parabola to write an equation for the parabola with directrix $y = c$ and focus (r, s), where c, r, and s are fixed, real numbers.

TEACHER'S GUIDE
HYPERBOLAS THROUGH PAPER-FOLDING

GOAL: To develop students' understanding of the definition of hyperbola and the significance of the foci and difference constant.

STUDENT OBJECTIVE:

✓ To construct hyperbolas through paper-folding and examine the effects of various placements of the foci.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity include knowledge of the definition of tangent lines, hyperbola, circle, and foci. We recommend that each student in the class be allowed to construct a hyperbola through paper-folding. When students begin to investigate the effects of varying the distance between foci and using various difference constants, we recommend that they work in cooperative groups, with each group member choosing a different set of parameters.

VOCABULARY: hyperbola, foci, circle, tangent line, equidistant, conic section

SUGGESTED PATH FOR REMEDIATION: If students have difficulty developing an intuitive feel for the definition of hyperbola after completing the paper-folding activity, we recommend the following course of remediation. Take one of the hyperbolas constructed through paper-folding, and select a point on the hyperbola. Using a ruler, measure the distance from this point to the focus inside the circle and the distance from this point to the focus outside the circle. The difference between these two distances should be equal to the radius of the circle. Have students select in turn several other points on the hyperbola and make like measurements. Seeing that the difference between the distance from the focus inside the circle to the selected point and the distance from the point to the focus outside the circle is the radius of the circle (independent of the selected points) should help to clarify the definition.

ADDITIONAL RESOURCES: Lines of symmetry for hyperbolas can be explored using a device called a Mira. Examples of using this device to locate lines of symmetry can be found in *Mira Math Activities for High School: A New Dimension in Motivation and Understanding*, a Mira Math Company publication.

HYPERBOLAS THROUGH PAPER-FOLDING

INTRODUCTION: A **hyperbola** has two U-shaped curves facing in opposite directions. A hyperbola is defined as the set of all points the difference of whose distances from two fixed points, called **foci**, is a constant. The conic section called hyperbola can be constructed from its definition by producing an envelope of tangents to the curve.

PURPOSES:

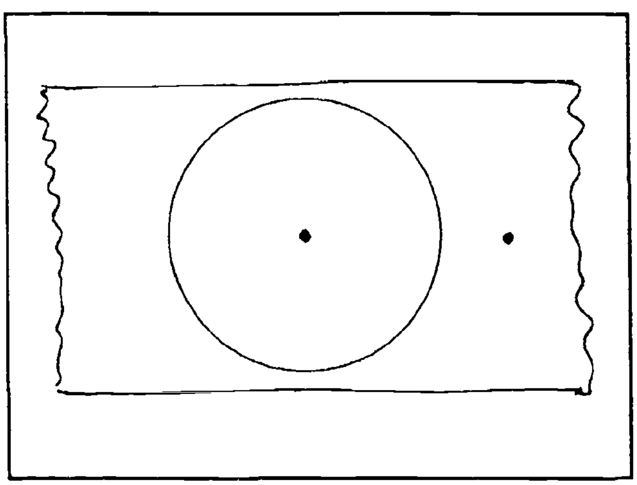
- ✓ Given a circle and a point outside the circle, how can paper-folding be used to construct the hyperbola that has the center of the given circle and the given point outside the circle as its two foci?
- ✓ How does varying the distance from the focus outside the circle to the circle change the resulting hyperbola?
- ✓ How does the radius of the given circle change the resulting hyperbola?

MATERIALS:

wax paper
felt tip marking pen
compass
several sheets of 8.5" X 11" construction paper
tape

PROCEDURES:

1. Tear off a piece of wax paper from 12 to 15 inches long. Use the felt tip pen to mark a point on the piece of wax paper. This point will serve as one of the foci for the hyperbola you will construct. Use the compass to draw a circle with the point you have marked as its center. The pencil in the compass will not make a distinct mark on the wax paper. You will need to use the felt tip pen to trace the circle so that it is easily visible. All the points on this circle are equidistant from the first focus. Use the felt tip pen to mark a second point outside the circle. This point will serve as the second focus of the hyperbola that you will construct.



8 - 1

2. Fold the paper so that the second focus lies over the circle. Crease the paper carefully. The line represented by this crease will contain a point on the hyperbola. Now fold the paper again to move the second focus to a different point over the circle. Crease again. Continue this process until the second focus has been moved all along the circle. The more creases (tangent lines), the sharper the outline of the hyperbola will be.
3. To help bring out the outline of the hyperbola enveloped by the creases (tangent lines), mount the wax paper on a piece of construction paper. To do this, lay the construction paper under the wax paper on which the paper-folding was done. Fold the edges of the wax paper under the edges of the construction paper, and tape it into place.
4. Use fresh pieces of wax paper and construction paper to perform the paper-folding activity several times, each time varying the placement of the foci and/or the length of the radius of the circle.

OBSERVATIONS:

1. What happens to the width of the hyperbola if the focus outside the circle is moved closer to the circle?

   It narrows.

2. What happens to the width of the hyperbola if the focus outside the circle is moved farther away from the circle?

   It widens.

3. What happens to the hyperbola if the radius of the circle is increased? Decreased?

   Increased branches are farther apart, decreased branches are closer together.

CONCLUSIONS:

1. Write a paragraph describing the relationship between the distance from the focus outside the circle to the circle and the width of the resulting hyperbola.

   As the focus outside the circle is moved closer to the circle, the resulting hyperbola has narrower branches. As the focus outside the circle is moved farther from the circle a hyperbola is produced with wider branches.

2. Write a paragraph describing the relationship between the radius of the circle and the resulting hyperbola.

The radius of the circle determines how far apart the two branches of the hyperbola are. The smaller the radius the closer the branches. The larger the radius the farther apart the branches.

SUGGESTIONS FOR FURTHER STUDY:

- Use the definition of hyperbola to write an equation for the hyperbola determined by the circle with center at (2, 4) and radius 3 and second focus, a point outside this circle (– 5, 2).
- Use the definition of a hyperbola to write an equation for the hyperbola determined by the circle with center (h, k) and radius r and the point outside the circle (u, v).

TEACHER'S GUIDE
ELLIPSES THROUGH PAPER-FOLDING

GOAL: To help students understand the definition of an ellipse and the significance of the foci and the summation constant.

STUDENT OBJECTIVE:

✓ To construct ellipses through paper-folding and examine the effects of various placements of the foci.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity include knowledge of the definitions of tangent line, ellipse, circle, and foci. We recommend that each student in the class be allowed to construct an ellipse through paper-folding. When students begin to investigate the effects of placing the foci various distances from each other and using various summation constants, we recommend that they work in cooperative groups, with each group member choosing a different set of parameters.

VOCABULARY: ellipse, focus (foci), circle, tangent line, equidistant, conic section

SUGGESTED PATH FOR REMEDIATION: If students have difficulty developing an intuitive feel for the definition of an ellipse after completing the paper-folding activity, we suggest the following activity. Take one of the ellipses constructed through paper-folding, and select a point on the ellipse. Using a ruler, measure the distance from the selected point to the focus at the center of the circle and the distance from the selected point to the other focus. The sum of these two distances should equal the radius of the circle. Have students select several more points on the ellipse and make like measurements. Discovering that the sum of the distance from the focus at the center of the circle to various selected points and the distance from these points to the other focus is the radius of the circle should help students see that the sum does not depend on the point selected.

ADDITIONAL RESOURCES: Lines of symmetry for ellipses can be explored using a device called a Mira. Examples of using this device to locate lines of symmetry can be found in *Mira Math Activities for High School: A New Dimension in Motivation and Understanding*, a Mira Math Company publication.

ELLIPSES THROUGH PAPER-FOLDING

INTRODUCTION: An **ellipse** is an oval-shaped curve defined as the set of all points the sum of whose distances from two fixed points, called foci, is a constant. ("Foci" is the plural of the word "focus.") You can construct the conic section called ellipse from its definition by producing an envelope of tangent lines to the curve as you fold paper by a prescribed technique.

PURPOSES:

✓ Given a circle and a point inside the circle other than its center, how can you use paper-folding to find the ellipse that has the center of the given circle and the given point inside the circle as its two foci?

✓ How does varying the distance from the focus at the center of the circle to the other focus alter the resulting ellipse?

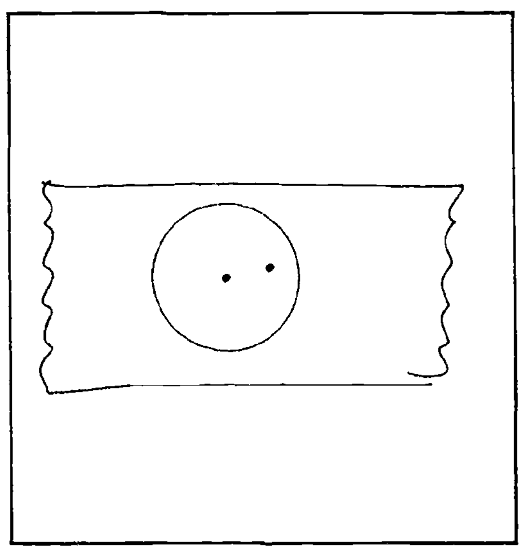
✓ How does varying the radius of the given circle change the resulting ellipse?

MATERIALS:

wax paper
felt tip marking pen
compass
several sheets of 8.5" X 11" construction paper
tape

PROCEDURES:

1. Tear off a piece of wax paper about 12 to 15 inches long. Use the felt tip pen to mark a point on the piece of wax paper. This point will serve as one of the foci for the ellipse you will construct. Use the compass to draw a circle with the point you have marked as its center. The pencil in the compass will not make a clear mark on the wax paper. You will need to use the felt tip pen to trace the circle so that it is easily visible. All the points on this circle are equidistant from the first focus, the center. Use the felt tip pen to mark a point inside the circle distinct from the center. This point will serve as the second focus of the ellipse that you will construct.



9-1

2. Fold the paper so that the second focus is over the circle you traced. Crease the paper carefully. The line represented by this crease will contain one point of the ellipse. Now move the second focus to a different point on the circle. Crease again. Continue this process until the second focus has been moved all along the circle. The more creases (tangent lines), the sharper the outline of the ellipse will be.
3. To help bring out the ellipse outlined by the creases (tangent lines), lay a piece of construction paper under the wax paper on which the paper-folding was done. Fold the edges of the wax paper under the edges of the construction paper, and tape the folds to the construction paper to hold the wax paper in place.
4. Use fresh pieces of wax paper and construction paper to perform the paper-folding activity several times, each time varying the placement of the foci and/or the length of the radius of the circle.

OBSERVATIONS:

1. How is the ellipse changed if the second focus is moved closer to the focus at the center of the circle?

   The closer the two foci are the more near circular is the ellipse that is produced.

2. How is the ellipse changed if the second focus is moved farther away from the focus at the center of the circle?

   The farther apart the foci the more elongated is the ellipse that is produced.

3. How is the ellipse changed if the radius of the circle is increased? Decreased?

   As the radius increases the ellipse becomes larger. As the radius decreases the ellipse becomes smaller.

CONCLUSIONS:

1. Write a paragraph describing how the relationship between the distance from one focus to the other effects the ellipse.

   The closer the foci, the more the ellipse resembles a circle.

2. Write a paragraph describing how varying the radius of the circle effects the ellipse.

   The larger the radius the larger the ellipse. The radius is the sum of the distances from the foci to the points on the ellipse.

SUGGESTIONS FOR FURTHER STUDY:

- Use the definition of ellipse to write the equation for the ellipse determined by the circle with center at (2, 4) and radius 3 and second focus the point inside this circle (3, 3).
- Use the definition of ellipse to write the equation for the ellipse determined by the circle with center (h, k) and radius r and the point inside the circle (u, v).

MEASUREMENT

TEACHER'S GUIDE
von KOCH SNOWFLAKE

GOAL: To develop students' understanding of sequences and series.

STUDENT OBJECTIVES:

- ✓ To derive a sequence representing the number of sides in consecutive iterations of the von Koch Snowflake.
- ✓ To derive a sequence representing the length of each side in consecutive iterations of the von Koch Snowflake.
- ✓ To derive a sequence representing the perimeter of consecutive iterations of the von Koch Snowflake.
- ✓ To derive a sequence representing the area of each new triangle in consecutive iterations of the von Koch Snowflake.
- ✓ To derive a sequence representing the new area in each consecutive iteration of the von Koch Snowflake.
- ✓ To derive a series representing the area of the von Koch Snowflake after an infinite number of consecutive iterations.
- ✓ To examine the limits of these sequences and series.

GUIDE TO THE INVESTIGATIONS: Students must have had previous instruction regarding sequences and series. They should know the Pythagorean Theorem and the formula for the area of a triangle. Experience with geometric sequences and series is particularly important. Students need to know that geometric sequences with a common ratio less than one converge to zero, while those with common ratio greater than one diverge. Similarly, students need to know that geometric series have a finite sum if and only if the common ratio is less than one. They should know the formula for finding the sum of an infinite geometric series with a ratio less than one, $S = \frac{a}{1 - r}$.

Students may work individually or in cooperative groups to carry out this investigation.. Students should be encouraged to measure carefully as they construct graphic representations of the first three iterations of the von Koch Snowflake. These may be displayed in the classroom or other appropriate area in school, along with charts showing the results of additional iterations.

After completing the investigation, discuss the unique nature of the von Koch Snowflake. It is the number of sides in the figure and the perimeter which grow without bound, while the length of each side and the area are bounded above.

VOCABULARY: sequence, series, iteration, equilateral triangle, trisect, perimeter, area, Pythagorean Theorem

SUGGESTED PATH FOR REMEDIATION: Students who have trouble with finding formulas for sequences may be aided by recording not only the numerical answers for each iteration but also their factorization. For example, the sequence which represents the number of sides in successive iteration is: 3, 12, 48, 192, 768,.... Many students see that the common ratio is 4 but still cannot write a general formula for the n^{th} term. Suggest that they write each term of the sequence as a product of their prime factors. This makes it a little easier to see that the general form is $(4^n)(3)$, where n is a natural number. Similar factorizations are helpful in developing the formulas for the other sequences and series in this investigation.

ADDITIONAL RESOURCES: The HiMAP Module 2, *Recurrence Relations*, provides additional materials for investigation. These materials are available from COMAP, Inc.

VON KOCH SNOWFLAKE

INTRODUCTION: The von Koch Snowflake is a sequence of figures beginning with an equilateral triangle. The second figure in the sequence is formed by trisecting each side of the triangle and constructing 3 new equilateral triangles using the center segment of each side as the base and then erasing it. This produces a six point star. To produce the third figure in the sequence, each of the 12 sides of the second figure is trisected and a new equilateral triangle is formed using the center segment of each side as a base, which is then erased. To produce the $(n+1)^{st}$ figure in the sequence, each of the sides of the n^{th} figure is trisected and a new equilateral triangle is formed using the center segment of each side as a base, and then the center segments from the sides of the n^{th} figure are erased.

PURPOSES:

- ✓ Can the first three figures in the von Koch Snowflake sequence be drawn using a ruler and protractor?
- ✓ Can a formula be found for the sequence representing the number of sides of each figure in the progression of von Koch Snowflakes?
- ✓ Can a formula be found for the sequence representing the length of each side of the figures in the progression of von Koch Snowflakes?
- ✓ Can a formula be found for the sequence of perimeters of the figures in the progression of von Koch Snowflakes?
- ✓ Can a formula be found for the sequence representing the area of each new triangle in the progression of von Koch Snowflakes?
- ✓ Can a formula be found for the sequence representing the total new area added in each iteration of the von Koch Snowflake sequence?
- ✓ Can a general formula be found for the series representing the total area of each figure in the progression of von Koch Snowflakes?

MATERIALS:

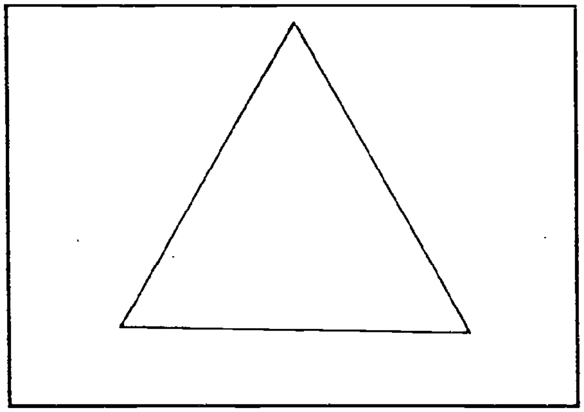
poster board
ruler
protractor
pencil
eraser
markers
calculator

PROCEDURES:

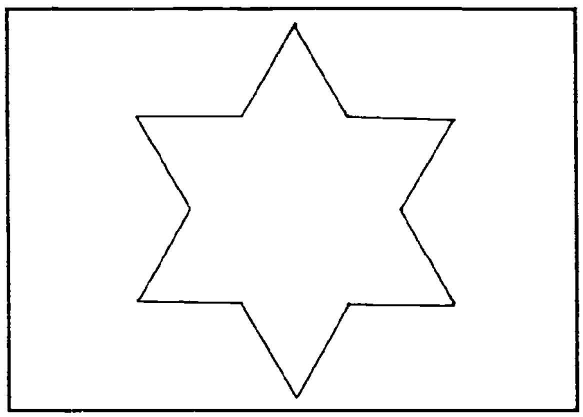
1. Use a pencil, ruler and protractor to draw an equilateral triangle with sides measuring 9 inches on each of three pieces of poster board. At the top of these posters write the words "start," "first iteration," and "second iteration." On the poster labeled "start", the triangle can be permanently drawn with a marker.

2. On the poster labeled "first iteration," mark each side into three equal parts using your ruler and pencil. Each section will measure 3 inches. With the center third of each side as a base, use your pencil, ruler, and protractor to draw three new equilateral triangles with sides of 3 inches each. Erase the center sections from each of the three original sides. This should produce a 6 point star, a twelve-sided polygon. Trace around the perimeter of this figure with a permanent marker.

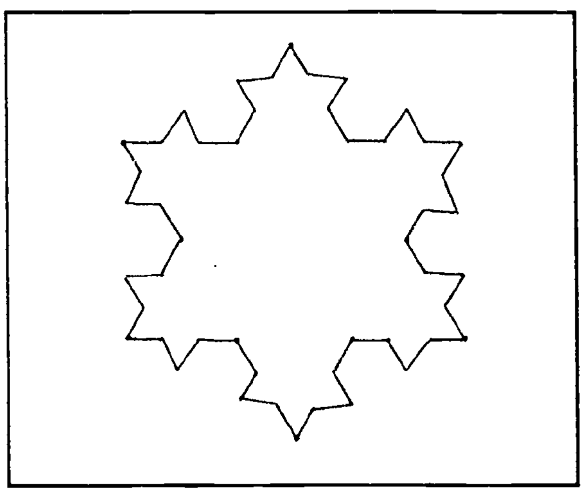
3. On the poster labeled "second iteration," repeat step two, but do not trace the perimeter with the marker. Using your pencil and ruler, mark each of the twelve sides into three equal parts. Each section will measure 1 inch. With the center third of each of the twelve sides as the base, use your pencil, ruler, and protractor to draw twelve new equilateral triangles with sides of 1 inch each. Erase the center section from each of the 12 sides to produce a 48 sided figure. Trace around the perimeter of this figure with a permanent marker.



10 - 1 *Start*



10 - 2 *First iteration*



10 - 3 *Second iteration*

OBSERVATIONS:

1. Use the Pythagorean Theorem to find the height of the first triangle. This will be an irrational number. You may use your calculator to estimate the height if you wish, but record the actual value of the height in the table provided. Fill in the table for starting triangle ($n = 0$).
2. Fill in the information on the table for the figure in the first iteration ($n = 1$).
3. Fill in the information on the table for the figure in the second iteration ($n = 2$).

iterations n	sides number	length	perimeter length	number of new triangles	height of new triangles	area of new triangles	total new area	total area
0	3	9	27	1	$\frac{9\sqrt{3}}{2}$	$\frac{81\sqrt{3}}{4}$	$\frac{81\sqrt{3}}{4}$	$\frac{81\sqrt{3}}{4}$ ≈ 35.07
1	12	3	36	3	$\frac{3\sqrt{3}}{2}$	$\frac{9\sqrt{3}}{4}$	$\frac{27\sqrt{3}}{4}$	$27\sqrt{3}$ ≈ 46.77
2	48	1	48	12	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{4}$	$3\sqrt{3}$	$30\sqrt{3}$ ≈ 51.96
3	192	$\frac{1}{3}$	64	48	$\frac{\sqrt{3}}{6}$	$\frac{\sqrt{3}}{36}$	$\frac{4\sqrt{3}}{3}$	$\frac{94\sqrt{3}}{3}$ ≈ 54.27
4	768	$\frac{1}{9}$	$\frac{256}{3}$	192	$\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{324}$	$\frac{16\sqrt{3}}{27}$	$\frac{862\sqrt{3}}{27}$ ≈ 55.30
5	3 072	$\frac{1}{27}$	$\frac{1\,024}{9}$	768	$\frac{\sqrt{3}}{54}$	$\frac{\sqrt{3}}{2\,916}$	$\frac{64\sqrt{3}}{243}$	$\frac{7\,822\sqrt{3}}{243}$ ≈ 55.75
6	12 288	$\frac{1}{81}$	$\frac{4\,096}{27}$	3 072	$\frac{\sqrt{3}}{162}$	$\frac{\sqrt{3}}{26\,244}$	$\frac{256\sqrt{3}}{2\,187}$	$\frac{70\,654\sqrt{3}}{2187}$ ≈ 55.96
7	49 152	$\frac{1}{243}$	$\frac{16\,384}{81}$	12 288	$\frac{\sqrt{3}}{486}$	$\frac{\sqrt{3}}{236\,196}$	$\frac{1\,024\sqrt{3}}{19\,683}$	$\frac{636\,910\sqrt{3}}{19\,683}$ ≈ 56.05
8	19 6608	$\frac{1}{729}$	$\frac{65\,536}{243}$	49 152	$\frac{\sqrt{3}}{1\,458}$	$\frac{\sqrt{3}}{2\,125\,764}$	$\frac{4\,096\sqrt{3}}{177\,147}$	$\frac{5\,736\,286\sqrt{3}}{17\,7147}$ ≈ 56.09
9	786 432	$\frac{1}{2\,187}$	$\frac{262\,144}{729}$	196 608	$\frac{\sqrt{3}}{4\,374}$	$\frac{\sqrt{3}}{19\,131\,876}$	$\frac{16\,384\sqrt{3}}{1\,594\,323}$	$\frac{51\,642\,958\sqrt{3}}{1\,594\,323}$ ≈ 56.10
10	3 145 728	$\frac{1}{6\,561}$	$\frac{1\,048\,576}{2187}$	786 432	$\frac{\sqrt{3}}{13\,122}$	$\frac{\sqrt{3}}{172\,186\,884}$	$\frac{65\,536\sqrt{3}}{14\,348\,907}$	$\frac{464\,852\,158\sqrt{3}}{14\,348\,907}$ ≈ 56.11

iterations n	sides number	length	perimeter length	number of new triangles	height of new triangles	area of new triangles	total new area	total area
n	$3 \cdot 4^n$	$\frac{9}{3^n}$ or $\frac{1}{3^{n-2}}$	$27\left(\frac{4}{3}\right)^n$ or $\frac{4^n}{3^{n-3}}$	$3 \cdot 4^{n-1}$ for $n \geq 1$	$\frac{9\sqrt{3}}{2}\left(\frac{1}{3}\right)^n$ or $\frac{\sqrt{3}}{2 \cdot 3^{n-2}}$	$\frac{81\sqrt{3}}{4(3^{2n})}$ or $\frac{81\sqrt{3}}{4 \cdot 9^n}$ or $\frac{\sqrt{3}}{4 \cdot 9^{n\ 2}}$	$\frac{3.4^{n-1} \cdot 81\sqrt{3}}{4.3^{2n}}$ or $\frac{243\sqrt{3}}{16}\left(\frac{4}{9}\right)^n$	$\frac{81\sqrt{3}}{4} + \frac{243\sqrt{3}}{16}\sum_{i=1}\left(\frac{4}{9}\right)^n$ or $\frac{81\sqrt{3}}{4} + \frac{243\sqrt{3}}{16} \cdot \frac{4}{5}\left(1 - \left(\frac{4}{9}\right)^n\right)$ or $\frac{81\sqrt{3}}{4} + \frac{243\sqrt{3}}{20}\left(1 - \left(\frac{4}{9}\right)^n\right)$ or $\frac{162\sqrt{3}}{5} - \frac{243\sqrt{3}}{20}\left(\frac{4}{9}\right)^n$ for $n \geq 1$
100	$3 \cdot 4^{100}$	$\frac{1}{3^{98}}$	$\frac{4^{100}}{3^{97}} \approx 8.42 \times 10^{13}$	$3 \cdot 4^{99}$ 1.21×10^{60}	$\frac{\sqrt{3}}{2 \cdot 3^{98}}$ 1.51×10^{-47}	$\frac{\sqrt{3}}{4 \cdot 9^{98}} \approx 1.32 \times 10^{-94}$	$\frac{243\sqrt{3}}{16}\left(\frac{4}{9}\right)^{100} \approx 1.59 \times 10^{-34}$	≈ 56.12
1000	$3 \cdot 4^{1\,000}$	$\frac{1}{3^{998}}$	$\frac{4^{1\,000}}{3^{997}}$	$3 \cdot 4^{999}$	$\frac{\sqrt{3}}{2 \cdot 3^{998}}$	$\frac{\sqrt{3}}{4 \cdot 9^{998}}$	$\frac{243\sqrt{3}}{16}\left(\frac{4}{9}\right)^{1\,000}$	≈ 56.12
10,000	$3 \cdot 4^{10\,000}$	$\frac{1}{3^{9\,998}}$	$\frac{4^{10\,000}}{3^{9\,997}}$	$3 \cdot 4^{9\,999}$	$\frac{\sqrt{3}}{2 \cdot 3^{9\,998}}$	$\frac{\sqrt{3}}{4 \cdot 9^{9\,998}}$	$\frac{243\sqrt{3}}{16}\left(\frac{4}{9}\right)^{10\,000}$	≈ 56.12

CONCLUSIONS:

1. Without drawing the figures, calculate and fill in the information in the table for the third through tenth iterations.
2. Write a general expression (formula) for each column in the chart. Record these in the indicated area on the chart.
3. Use the formulas and a calculator to find decimal approximations for each of the values in the table for the 100th iteration, the 1000th iteration, the 10,000th iteration.
4. Describe in your own words what is happening to the number of sides as the number of iterations increases; what is happening to the perimeter as the number of iterations increases; what is happening to the area as the number of iterations increases? *sides → ∞; perimeter → ∞; area → ≈ 56.12*

SUGGESTIONS FOR FURTHER STUDY:

- This investigation uses the fact that geometric sequences with a common ratio less than 1 converge. Research other types of convergent sequences.
- This investigation uses the fact that infinite geometric series with a common ratio less than 1 have a finite sum. Find other infinite series with finite sums.

TEACHER'S GUIDE ERRONEOUS MEASUREMENT

GOAL: To develop students' understanding of the concept that error in measurement is often compounded when erroneous measures are used in calculations.

STUDENT OBJECTIVES:

- ✓ To make simple linear measurements and calculate the error for each measurement.
- ✓ To use the measured linear dimensions of rectangular prisms to calculate their volumes.
- ✓ To find the error in volume resulting from the error in linear measurement.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are the ability to measure to the nearest millimeter using a metric ruler and knowledge of percentage, mean, and standard deviation. Students who are already familiar with these concepts may want to use a calculator to perform the computations.

This activity can be conducted by students working in cooperative groups or as individuals. It requires that each group or student ask ten different people to measure the length, width, and height of a rectangular prism. This can be accomplished by students making measurements for each other or by allowing students time outside of class to request that friends and/or family members make the measurements. The rectangular solids for this activity can be blocks of wood or household objects such as detergent boxes, shoe boxes, etc.

VOCABULARY: millimeter, mean, length, width, height, standard deviation, square, square root, standard error of the mean, volume, percentage of error

SUGGESTED PATH FOR REMEDIATION: Some students may need to practice measuring to the nearest millimeter. You may need to assist them as they make the first few measurements or even do an example with a line drawn or projected on the board using a clearly marked meter stick. Calculators may be useful with this activity.

ADDITIONAL RESOURCES: *Quantitative Literacy Series: Exploring Data*, by James M. Landwehr and Ann E. Watkins (Dale Seymour Publications) is an excellent source of activities related to statistics.

ERRONEOUS MEASUREMENT

INTRODUCTION: Measurement is never an exact undertaking. Even though many machine parts are milled to 0.001-inch tolerances and the microchips which drive computers are manufactured to ever smaller tolerances, there is *always* a margin of error for every measurement. Any error in initial measurement is likely to be compounded when erroneous measures are used in calculations.

PURPOSES:

✓ If ten different people measure the same linear distance, how can you mathematically define error in measurement?

✓ When erroneous measurements are used in calculations how can the error in measurement effect the accuracy of the result of the calculations?

MATERIALS:

metric ruler
rectangular shaped objects
calculator (optional)

PROCEDURES:

1. Select a rectangular prism from the collection provided. If it is deformable, take care that all the handling will not distort its measurements.
2. Have ten different people measure the length, width, and height of this rectangular prism. Record these measures in the chart provided below. Calculate the volume using each person's measurements.

measurement no.	length	width	height	volume
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

3. Determine the mean length, L_{mean}, by adding the ten length measurements and dividing the sum by ten.

4. Determine the deviation from the mean d for each length measurement by subtracting the measure from the mean. Record these in the table below. Be sure to indicate the signs, positive if the measurement is more than the mean; negative, if less.

measurement no.	length	mean length	difference
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

5. Determine the mean deviation, d_{mean}, from the mean by adding the *absolute values* of the deviations and dividing their sum by ten.

6. Determine the standard deviation for the length measurements σ_l by adding the squares of the deviations and dividing the sum of the squares by ten and then taking the square root of that quotient.

7. Determine the standard error of the mean for the length S_{ml} by dividing the standard deviation by the square root of one less than the number of measurements $\sqrt{(10-1)} = 3$. The actual length of the rectangular prism lies within the range of the mean length plus or minus the standard error of the mean.

8. Make similar tables to repeat this process for the width and height of the rectangular prism, and the volumes calculated using these measures. You will need to find W_{mean}, S_{mw}, H_{mean}, and S_{mh} .

9. Now find the estimated volume of the rectangular prism by multiplying the mean measured length times the mean measured width times the mean measured height.

10. The error in the latter volume calculation is determined by combining the percentage of errors in the length, width, and height to obtain the percentage of error for the volume. The formula below will give the percent of error for the volume.

$$\text{Vol error} = 100\sqrt{\left(\left(\frac{S_{ml}}{L_{mean}}\right)2 + \left(\frac{S_{mw}}{W_{mean}}\right)2 + \left(\frac{S_{mh}}{H_{mean}}\right)2\right)}$$

OBSERVATIONS:

1. How do the standard error of the mean for the length, width, and height compare?

Answers will vary depending on the observations as well as the measurements.

All of these should be about the same.

2. How do each of the standard errors of the means for the ten volumes compare with the standard errors of the corresponding means for the length, width, and height?

Standard error of mean for volume is much greater than for linear measure.

3. How does the mean of the ten volumes calculated using the measurements compare with the volume calculated using the mean length, mean width, and mean height?

They should be about the same.

CONCLUSIONS:

1. A grain elevator worker measures the length, width, and height of a grain bin and uses this information to calculate the volume of the bin. He orders grain to fill the bin based on this volume calculation. Is it possible that he may not order the right amount of grain? Why or why not?

   Errors in measurement can result in errors in the calculation of volume.

2. List some examples of situations in which measurement error might need to be considered when doing calculations.

   Answers vary but may include things such as: the surface area of a house to be painted, the volume of a gasoline tank, the capacity of a storage space.

SUGGESTIONS FOR FURTHER STUDY:

- Consider the effect of the measurement error on computation of the surface area of the rectangular solid.
- Conduct a similar activity using cylinders, pyramids, or other three-dimensional shapes. What are the effects of measurement errors on the calculation of volumes and surface areas for these other geometric solids?

TEACHER'S GUIDE
HOW TALL IS THE SCHOOL FLAGPOLE?

GOAL: The student will develop an understanding of how trigonometry can be used to find measures that cannot be taken in traditional ways.

STUDENT OBJECTIVES:

- ✓ To form a set of similar triangles.
- ✓ To use trigonometry and given measures to find missing measures for the set of similar triangles.
- ✓ To compare the calculated measures with actual measures.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this investigation are a knowledge of trigonometric functions and similar triangles. Students should work in cooperative groups to collect and analyze the data from this investigation. (The data collection procedure requires more than one person.)

VOCABULARY: triangle, similar triangles, trigonometric functions, angle, side, angle side angle (ASA) postulate

SUGGESTED PATH FOR REMEDIATION: This is an excellent activity for students who are having difficulty with problem solving in introductory trigonometry because it provides an opportunity for them to develop insight through experience. Students may need some review of basic geometry, e. g., the sum of the measures of the interior angles of a triangle.

ADDITIONAL RESOURCES: *Sourcebook of Applications of School Mathematics*, prepared by the Joint Committee of the Mathematical Association of America and National Council of Teachers of Mathematics, contains a section, "Trigonometry and Logarithms," which includes similar investigations.

HOW TALL IS THE SCHOOL FLAGPOLE?

INTRODUCTION: Many trigonometry texts ask you to find the height of a flagpole given the length of its shadow and the measure of an angle of inclination. In this activity you will explore why the procedure used to solve this type of problem works.

PURPOSES:

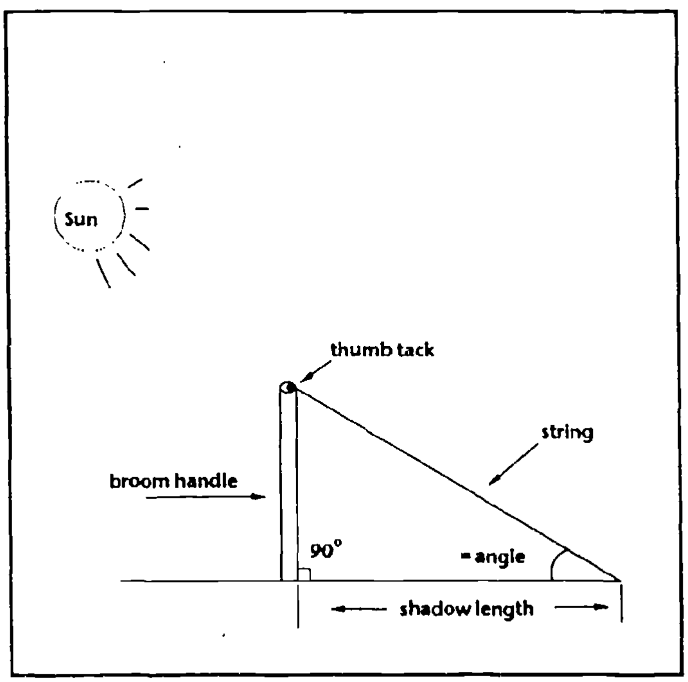
- ✓ What is an angle of inclination?
- ✓ Are similar triangles really a help?

MATERIALS:

a meter stick
a thumb tack
a protractor
a sunny day
a broom handle
a piece of string longer than a broom handle.
scissors
construction paper

PROCEDURES:

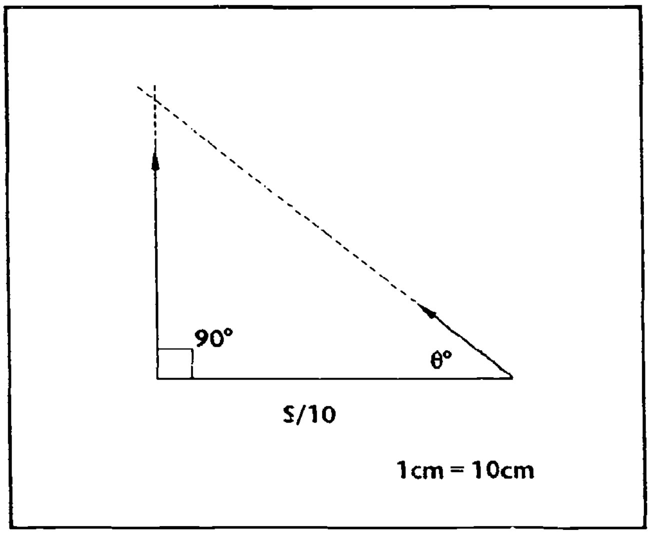
1. Take the meter stick, thumb tack, string, scissors, protractor, and broom handle out into the sun. Use a thumb tack to fasten the string to the top of the broom handle. One student should stand, holding the broom handle vertically so that its shadow can clearly be seen. Be sure the broom handle is perpendicular with the ground. Another student should match the string with the end of the shadow of the broom handle and hold it there, tightly but without stretching it, so that it can be cut to this length. Use



12-1

the protractor to measure the angle formed with the ground and the string at the point that is the end of the broom's shadow. Use the meter stick to measure the length of the shadow of the broom stick. Do not measure the length of the broom handle or the length of the string. Record the angle measure and the length of the shadow in the blanks provided in the Observation section.

2. Return to the classroom. On construction paper, make a scale model of the triangle you formed outside. Draw a line segment representing the shadow using the scale that you have chosen. For example, if the shadow measures 55 cm and the scale is 1 cm to 10 cm, then you would draw a line segment 5.5 cm long. On one end of this segment, draw a 90° angle. On the other end draw an angle whose measure is equal to the measure of the angle made by the string and the shadow of the broom handle. Once these two angles have been drawn, sketch in the full triangle. Cut this triangle out of the construction paper. Measure the other two sides of this triangle, and record the measures in the observation section of this investigation.



12 - 2 Scale of 1 to 10

OBSERVATIONS:

1. Record the length of the shadow of the broom handle and the angle measure of the angle formed by the shadow and the string.

shadow length (cm) = ____________

angle between shadow & string (deg) = ____________

2. Record the measures of sides and angles of the construction paper triangle.

side representing shadow (cm) = ____________

side representing string (cm) = ____________

side representing broom handle (cm) = ____________

angle between shadow & string (deg) = ____________

angle between shadow & broom handle (deg) = ____________

angle between broom handle & string (deg) = ____________

CONCLUSIONS:

1. You know that the sum of the measures of the angles of a triangle is 180° and the measure of the angle of the broom handle with the shadow is 90°. You have recorded the measure of the angle between the shadow and the broom handle, so what is the measure of the angle of the broom handle and the string?

   90° less your measurement.

2. Is the scale model you constructed similar to the actual triangle you built outside with the broom handle, the string, and the shadow?

   Yes.

3. Use similar triangles and the appropriate trigonometric functions to calculate the length of the broom handle and the length of the string used for the third side of the triangle.

   Answers will vary.

4. Measure the broom handle and the string. How do these measurements compare with the answers you arrived at using similar triangles?

   Answers will vary.

SUGGESTIONS FOR FURTHER STUDY:

- Build and use a clinometer to construct scale models for real life similar triangle problems. For example, find the height of the flag pole at your school, the height of a tall building, and the height of a tree.
- Investigate other examples of similar triangles.

TEACHER'S GUIDE
PI ARE SQUARE!

GOAL: To develop students' understanding of the formula for the area of a circle.

STUDENT OBJECTIVE:

✓ To construct a model to demonstrate why the formula πr^2 gives the area of a circle.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are an understanding of the concept of area, a working understanding of the area formula for rectangles, an understanding of vocabulary related to circles, including the relationship between the circumference and diameter of a circle: $\pi = \frac{\text{circumference}}{\text{diameter}}$.

Each student or group will need a compass, construction paper, protractor, straight edge, scissors, and poster board. Have each follow the instructions given in the Procedures section to produce a poster showing the circle and the rearrangement of the circle. Each should complete the Observation and Conclusion sections of this activity. It is important to make the connection between this model and the concept of a limit. With the circle cut into sixteen equal pieces and rearranged, the resulting figure is not really a rectangle, but we believe that if we continued to cut the circle into more and more pieces, the rearrangement would get closer and closer to a rectangle without ever actually becoming one. We might say that, as the number of equal pieces approaches infinitely many, the rearrangement approaches a rectangle.

VOCABULARY: circle, radius, diameter, area, circumference, rectangle

SUGGESTED PATH FOR REMEDIATION: Many students do not have a conceptual understanding of area. This may not be evident from paper and pencil evaluations, because such students may have memorized formulas that allow them to perform computations and produce correct answers. One major purpose of this activity is to help these students develop better conceptual understanding. Circle geoboards may also be used to develop this concept. Their use provides one possible path for remediation for students having difficulty.

ADDITIONAL RESOURCES: If a circle is constructed on a square lattice, then Pick's Theorem can be used to find the area: $A = \frac{e}{2} + i - 1$, where i is the number of dots inside the circle and e is the number of dots on the circumference of the circle. This theorem provides for an interesting discussion of alternative formulas and methods and allows students to compare results from the different formulas.

PI ARE SQUARE!

INTRODUCTION: You probably learned the formula for finding the area of a circle in elementary school: π times the square of its radius. In this activity you are going to try to discover the origin of this ancient formula.

PURPOSE:

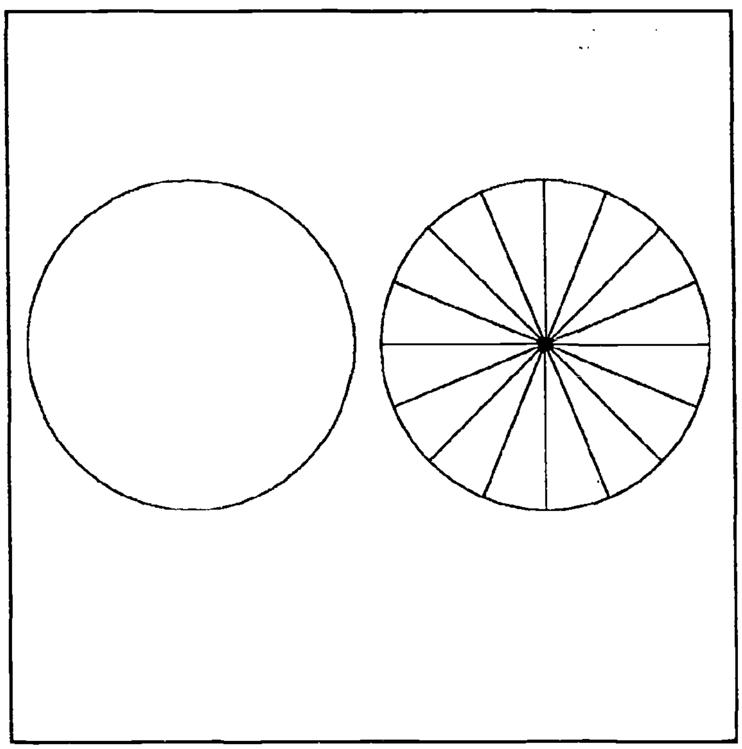
✓ How can you construct a model to demonstrate why the formula πr^2 gives the area of a circle?

MATERIALS:

construction paper
compass
protractor
scissors
straight edge
poster board

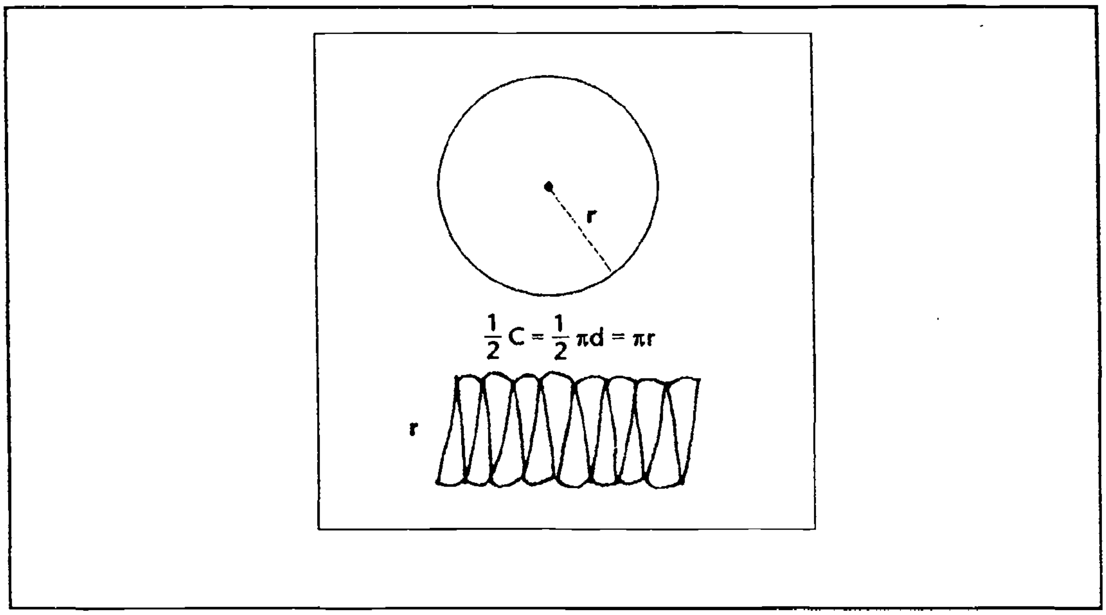
PROCEDURES:

1. Choose a point on a piece of construction paper to serve as the center of a circle. Use the compass to draw a large circle. Cut out this circle.
2. Trace your circle onto a second piece of construction paper. Cut out this circle, and mount it on the poster board.
3. Use the protractor and straight edge to draw eight diameters separating the original circle into sixteen equal sections. Cut along these diameters producing sixteen equal, wedge shaped pieces.



13 - 1 *Separate the circle into sixteen equal pieces.*

4. Rearrange these pieces as shown in illustration 13-2 and mount them on the poster board below the circle.



13-2

OBSERVATIONS:

1. Would you agree that the rearrangement of the circle looks somewhat like a rectangle?

   Answers vary: roughly, yes.

2. What is the width of this "rectangle" in terms of the radius of the original circle?

   It is equal to the radius.

3. What is the length of this "rectangle" in terms of the radius of the original circle?

   $\frac{1}{2}C$; *or* $\frac{1}{2}\pi d$; *or* πr

4. What is the area of this "rectangle" in terms of the radius of the original circle?

   $\pi r \cdot r$; *or* πr^2

CONCLUSIONS:

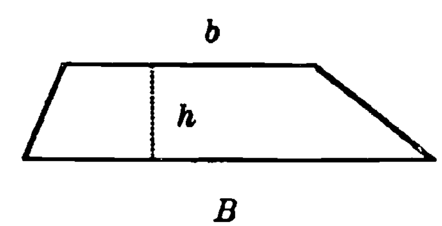
1. What will happen to the rearrangement of the circle if it is cut into thirty-two equal pieces instead of sixteen?

The measured area will more closely approximate the area of the circle.

2. What will happen to the rearrangement of the circle if it is cut into infinitely many equal pieces?

The area should equal the area of the circle.

3. The formula for the area of a trapezoid is $\frac{b + B}{2h}$. Construct a model to show how a trapezoid can be divided into two triangles such that the combined area of these two triangles produces the correct result.



13 - 3

SUGGESTIONS FOR FURTHER STUDY:

- Can this idea of rearrangement of a circle to produce a rectangle be extended into three dimensions? In other words, can a sphere be rearranged to produce a rectangular prism? Why or why not?

TEACHER'S GUIDE
TRISECTING ANGLES

GOAL: To develop students' understanding of the concept of trisection and to provide an application of the notion of congruent triangles.

STUDENT OBJECTIVE:

✓ To develop a tool which will allow you to trisect an angle.

GUIDE TO THE INVESTIGATIONS: Trisection of angles cannot be accomplished via traditional compass and straight edge construction; however, you can construct a tool to trisect an angle. The prerequisite skills for this activity are an understanding of congruences of line segments, angles, and triangles, methods of proving triangles congruent, definitions of bisection and trisection, constructing circles, constructing congruent line segments, constructing a perpendicular to a line through a point on the line, and measuring angles.

Each student or cooperative group should use the directions in the Procedures section to produce the trisection tool. In order to make this tool they will need a sheet of poster board or cardboard, scissors, a compass, and a ruler. Once the tool has been constructed, each student should follow the directions for using the tool to perform the trisection of an angle. After the angle has been trisected, students may complete the Observation and Conclusion sections of the activity.

VOCABULARY: circle, semicircle, radius, circumference, center, bisect, trisect, perpendicular, congruent, angle, vertex, tangent, coincides, proof

SUGGESTED PATH FOR REMEDIATION: Students who have difficulty with this activity may lack some of the requisite skills. Having them work in cooperative groups will help because other students in the group will share knowledge with them and enhance their own understanding. Some review may be necessary—students may not recall how to construct a perpendicular to a line through a given point on the line.

ADDITIONAL RESOURCES: A description of this trisection tool may be found in *A Handbook of Aids for Teaching Junior-Senior High School Mathematics*, by Stephen Krulik (W. B. Saunders Company, 1971).

TRISECTING ANGLES

INTRODUCTION: In this activity you will build a tool to trisect an angle.

PURPOSES:

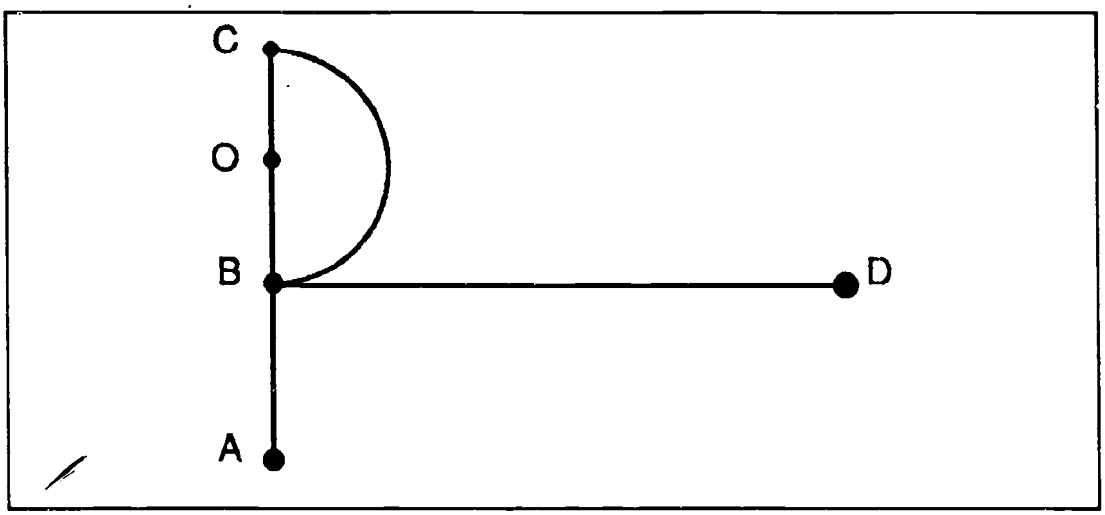
- ✓ How can you build a tool that can be used to trisect an angle?
- ✓ Can you show that this tool works?

MATERIALS:

heavy poster board or cardboard
scissors
compass
ruler
protractor

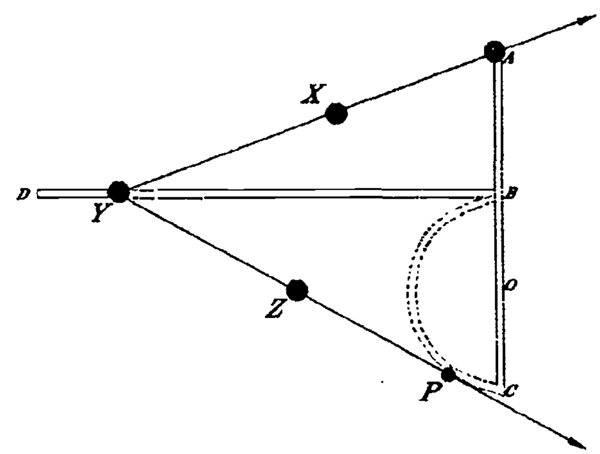
PROCEDURES:

1. On the cardboard draw a semicircle having an arbitrary radius. For trisecting angles drawn on regular paper, a radius of 2cm works well. Label the end points of the diameter of the semicircle *B* and *C*. Label the center of the semicircle *O*.
2. Extend the diameter *BC* to Point *A* so that the distance from *A* to *B* is equal to the radius of the circle and the points are ordered *A-B-O-C*. (See illustration 14–1.)
3. Construct a perpendicular to line *AC* at point *B*, label another point on the perpendicular *D*. Choose *D* so that the length of line segment *BD* is about five or six times the radius chosen for the semicircle. Note line segment *BD* is tangent to the semicircle. (See illustration 14–1 below.)



14 · 1

4. Cut out the instrument. On a piece of paper, draw an angle and label it XYZ, with Y the vertex. To use this instrument to trisect angle XYZ, place point A on side $\overrightarrow{XY}$ of the angle. Slide the device until line BD coincides with the vertex of the angle Y and the semicircle is tangent to side $\overrightarrow{YZ}$ of the angle. Call this point of tangency P. Lines YB and YO trisect angle XYZ.



14 - 2

OBSERVATIONS:

1. Use the protractor to find the measure of angle AYB.

   $= \frac{1}{3} m \angle XYZ$

2. Use the protractor to find the measure of angle BYO.

   $= \frac{1}{3} m \angle XYZ$

3. Use the protractor to find the measure of angle OYP.

   $= \frac{1}{3} m \angle XYZ$

4. Do these measurements indicate that the trisection was successful?

   Yes.

CONCLUSIONS:

1. Prove that triangles *AYB*, *OYB*, and *OYP* are congruent and thus line segment *BY* and *OY* trisect angle *XYZ*.

$\angle AYB$, $\angle OBY$ are right angles since $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$

$\overline{BY} \cong \overline{YB}$ *by reflexive property*

$\overline{AB} \cong \overline{OB}$ *(See construction of trisection tool.)*

$\Delta AYB \cong \Delta OYB$ *by HL*

$\angle OPY$ is a right angle since $\overleftrightarrow{YP}$ *is tangent to circle O.*

$\overline{OP} \cong \overline{OB}$ *since they are radii of the same circle.*

$\overline{OY} \cong \overline{OY}$ *by reflexive property*

$\Delta OYB \cong \Delta OYP$ *by HL.*

SUGGESTIONS FOR FURTHER STUDY:

- Combine the methods of bisecting angles and trisecting angles to divide a 90° angle producing an angle with measure 7.5°.

TEACHER'S GUIDE
PICK'S THEOREM

GOAL: The student will develop an understanding of a process used to estimate areas of irregular shapes.

STUDENT OBJECTIVES:

- ✓ To conduct a geoboard activity that will lead to the discovery of Pick's Theorem.
- ✓ To apply Pick's Theorem to approximate the area of a lake or park on a map of your state.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are an understanding of area and formulas for area as well as experience finding areas of polygons on the geoboard. This activity can be conducted individually or in cooperative groups. If students have not previously used a geoboard, training in introductory procedures for finding areas of polygons on the geoboard will be helpful.

Constructing the grid on acetate (overhead transparency sheet) can be difficult, especially if the ink from the marker runs. An easy way to avoid this problem is to have students construct the grid on a piece of white paper using a black pen, and then use a copying machine to make the transparency.

If you are having trouble locating a sufficient number of state maps for this activity, try contacting your state bureau of tourism, the local Chamber of Commerce, an Interstate Highway Welcome Center, or a State Patrol Post.

VOCABULARY: polygon, triangle, boundary point, interior point, geoboard, area, scale, approximate

SUGGESTED PATH FOR REMEDIATION: Some students are better at generalization than others. Some will have trouble writing a formula for area using the number of interior points and boundary points, so it may be necessary to give them the formula and then encourage them to check the formula against the results recorded from their investigation. It may also be necessary to review the use of scale drawings so that students will understand why it is important to use a grid marked off in squares whose side lengths represent one mile on the map.

ADDITIONAL RESOURCES: Teachers unfamiliar with geoboards may refer to *Learning with Geoboards*. This little pamphlet from Cuisinaire gives the basics of working with geoboards. There are also several resource books with geoboard activities available from educational publishers such as Creative Publications and Dale Seymour.

PICK'S THEOREM

INTRODUCTION: Situations frequently arise in which you need to approximate the area of an irregular shape. The purpose of this investigation is to develop and apply a method for doing so.

PURPOSES:

✓ Can a process be developed to approximate the area of irregular shapes?

✓ How can this process be applied to regions on maps?

MATERIALS:

geoboard
rubber bands
state map
clear overhead transparency
ruler
markers for transparencies

PROCEDURES:

1. On a geoboard construct a triangle with no pegs (points) in its interior. Find and record its area in the table provided in the Observations section. Next, construct a polygon having four boundary points and no interior points. Find and record its area on the given table. Repeat this process for figures with five and six boundary points, each with no interior pegs.
2. Construct a triangle having one peg in its interior. Find and record its area on the given table. Do likewise for polygons having four, five, and six boundary points, each with one interior point.
3. Repeat the process for polygons having three, four, five, and six boundary points and, in succession, two, three, and four interior points.
4. Complete question 1 in the Conclusion section.
5. Construct a grid on the overhead transparency using the marker and ruler. Make it so that each square on the grid has sides with the length used on the state map to represent one mile.
6. Choose a lake or park with irregular borders from the state map. Lay the grid you have constructed over the map so that the map area representing the lake or park is under the grid. Count the number of crossing points on the grid that lie on the boundary of the lake. Count the number of crossing points of the grid that lie in the interior of the area that represents the lake.
7. Complete number 2 in the Observation section.

8. Complete question 2 in the Conclusion section.

OBSERVATIONS:

1. Record information from the geoboard investigation on the table below.

Boundary Points	Interior Points	Area
3	0	½ $unit^2$
4	0	1 $unit^2$
5	0	3/2 $unit^2$
6	0	2 $units^2$
3	1	3/2 $units^2$
4	1	2 $units^2$
5	1	5/2 $units^2$
6	1	3 $units^2$
3	2	5/2 $units^2$
4	2	3 $units^2$
5	2	7/2 $units^2$
6	2	4 $units^2$
3	3	7/2 $units^2$
4	3	4 $units^2$
5	3	9/2 $units^2$
6	3	5 $units^2$
3	4	9/2 $units^2$
4	4	5 $units^2$
5	4	11/2 $units^2$
6	4	6 $units^2$

2. When the grid you constructed was laid over the area representing the lake or park, how many crossing points of the grid were lying on the boundary of the lake?

Answers vary.

3. How many crossing points of the grid were lying in the interior of the area representing the lake or park?

Answers vary.

CONCLUSIONS:

1. Based on the information from the table, can you write a formula using I for the number of interior points, B for the number of boundary points, and A for the area, to express A as a function of I and B?

   $\frac{1}{2}(B-2)+I=A$

2. The formula that you developed above is called Pick's Theorem after the person who developed it. Can you use this theorem to approximate the area in square miles of the lake or park you chose on the map?

   Answers will vary.

SUGGESTIONS FOR FURTHER STUDY:

- Use Pick's Theorem to approximate the area of the palm of your hand in square centimeters. Use a sheet of centimeter grid paper. Trace the outline of your hand on the paper, and follow the procedures you have been using.
- Investigate the history and applications of Pick's Theorem.

NUMBER SENSE

TEACHER'S GUIDE
DOES 0.9999... REALLY EQUAL 1?

GOAL: To help students understand that repeating decimals are rational numbers and to recognize results they may obtain with their calculators.

STUDENT OBJECTIVES:

- ✓ To explore patterns in the relationship between a repeating decimal and its fractional expression.
- ✓ To find the fractional equivalent of a repeating decimal.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are an understanding of rational numbers, repeating and nonrepeating decimals, the ability to use a calculator to find the decimal representation of a fraction, and some skill in looking for patterns. Repeating decimals are rational numbers. Thus each repeating decimal can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Students may work alone or in cooperative groups to perform this activity. Each student or group will need a calculator and several sheets of poster board to make charts. They should follow the instructions in the Procedures section. They should be encouraged to generalize the results of each of their explorations.

Suggestions for further study can be used to make the connection between algebraic methods of finding fractional representations for repeating decimals and the patterns discovered in this activity.

VOCABULARY: rational number, repeating decimal, numerator, denominator, fraction, equivalent, ellipsis

SUGGESTED PATH FOR REMEDIATION: When they see 0.1111... written $0.\overline{1}$, many students believe that it is equivalent to $\frac{1}{10}$. (That may be a good reason to abandon the latter notation in favor of the more expressive pattern of repetition followed by the ellipsis. We think that almost all students can tell from the notation 0.789123123123... what is repeating!) With increasing use of technology, it is more important than ever to help students develop number sense. Without an intuition for what they ought to get, students will often be unable to interpret a calculator's screen in any meaningful way. This activity tries to encourage students to use their experience to develop intuition about the relationship between repeating decimals and fractions. Some students may have difficulty with this approach because they are not comfortable with the guess-test-revise problem-solving strategy. When students are presented with a repeating decimal and asked to find its fractional equivalent, encourage them to guess, use the calculator to test the guess, and then make another, hopefully more accurate guess based on the results of their trial. In

order to make subsequent guesses that reflect information from previous guesses, they must be able to rank decimals in order of magnitude. If a student is having a great deal of trouble making second and third guesses, a review of ordering of decimals may be needed.

Encourage students to read decimals such as 0.123 not as "zero point one two three" but as "one hundred twenty-three one-thousandths." When they read *the number signified* rather than the literal symbols which represent the number, they stand a better chance of relating to it.

ADDITIONAL RESOURCES: Virtually every algebra textbook has a section on converting repeating decimals to fractions and vice-versa.

DOES 0.9999... REALLY EQUAL 1?

INTRODUCTION: Since you have come to rely so heavily on calculators, it is probably more important now than before to be able to look at a decimal representation of a number and relate it to its fractional name. If the calculator screen displays .33333333, then the answer may be $\frac{33333333}{100000000}$ or $\frac{1}{3}$. These two numbers are not equal even if they are close. If you decide that a calculator display of .33333333 really means 0.33333333..., a repeating decimal, then how can you find its fractional "name," that is, the fraction to which it is equal? That is what this activity is designed to help you discover. This will be especially important if you are required to give an exact answer in rational form rather than a repeating decimal. (While one-third is easy to recognize in decimal form, five-thirteenths may not be.)

PURPOSES:

- ✓ How can you find a fractional "name" for a repeating decimal?
- ✓ When are repeating decimals equivalent to terminating decimals?

MATERIALS:

calculator
poster board
markers

PROCEDURES:

1. On a sheet of poster board, construct a chart with two columns, one headed fraction; the other, decimal.
2. Write $\frac{1}{9}$ in the fraction column. Use your calculator to compute 1 divided by 9. Since your calculator display shows 0.11111111, you may conclude that $\frac{1}{9}$ is equivalent to 0.11111111.... You may divide one by nine by hand to confirm this fact. Record the decimal form in the column headed decimal.
3. Repeat this process for $\frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}$, and $\frac{8}{9}$.
4. On another sheet of poster board, make a second chart with two columns, and use this chart to explore decimal equivalents for $\frac{1}{99}, \frac{2}{99}, \frac{3}{99}$, and so on.
5. On another sheet of poster board, make a third chart with two columns, and use this chart to explore decimal equivalents for $\frac{1}{999}, \frac{2}{999}, \frac{3}{999}$, and so on.

6. Use charts to explore decimal equivalents for fractions with 90, 990, and 9990 as their denominators.

7. Use charts to explore decimal equivalents for fractions with denominators of 900, 9900, and 99900 as their denominators.

OBSERVATIONS:

1. In the first chart you made containing fractions with denominator of 9, do you recognize a pattern?

   There is a one digit repeating block of numbers. The repeating block is the numerator.

2. If this pattern holds, what is the fraction "name" for 0.999999999...?

   $^9/_9$ or 1

3. If a fraction has 99 as its denominator, how many digits are in the repeating block when this fraction is written as a decimal?

   Two digits

4. If a fraction has 999 as its denominator, how many digits are in the repeating block when this fraction is written as a decimal?

   Three digits

5. How many digits after the decimal point does the repeating block begin in decimal representations of fractions with 9, 99, or 999 as their denominator?

   The block begins with the first digit.

6. How many digits after the decimal point does the repeating block begin in the decimal representations of fractions with 90 as their denominator?

   The block begins with second digit.

7. How many digits long is the repeating block in decimal representations of fractions with 90 as their denominator?

   The block is one digit.

8. How many digits after the decimal point does the repeating block begin in decimal representations of fractions with 990 as their denominator?

   The block begins with the second digit after the decimal point.

9. How many digits after the decimal point does the repeating block begin in decimal representations of fractions with 900 as their denominator?

   The block begins with the third digit after the decimal point.

10. How many digits long is the repeating block in decimal representations of fractions with 900 as their denominator?

One digit

CONCLUSIONS:

1. What is the denominator of the fractional representation of 0.23777777...?

$900 = (10^1 - 1)(10^2)$

2. Find the numerator of the fraction in #1 by using the guess, test, and revise strategy. Use the table to help organize your guessing.

guess numerator	denominator	decimal	target	next number guess range
237	900	0.263333...	0.237777...	less than 237
207	900	0.23	0.237777...	between 207 & 237
220	*900*	*0.24444*	*0.237777...*	*between 207 & 220*
215	*900*	*0.23888*	*0.237777...*	*between 207 & 215*
210	*900*	*0.23333*	*0.237777...*	*between 210 & 215*
212	*900*	*0.23555*	*0.237777...*	*between 212 & 215*
213	*900*	*0.23666*	*0.237777...*	*between 213 & 215*
214	*900*	*0.23777*	*0.237777...*	*answer $^{214}/_{900}$ or $^{107}/_{450}$*

3. What is the denominator of the fractional representation of 0.345345345...?

999

4. Find the numerator of the fraction in #3 by using the guess, test, and revise strategy. Use the table following this activity to help organize your guessing.

guess numerator	denominator	decimal	target	next number guess range
345	*999*	*.345345...*	0.345345...	*answer* $^{345}/_{999}$ *or* $^{115}/_{333}$

5. Write a rule for determining the denominator of the fractional representation of any repeating decimal.

If the decimal simply repeats in blocks of n digits, then the denominator is $10^n - 1$.

(1 digit = 9; 2 digits = 99; 3 digits = 999 etc.)

If the decimal begins with m non-repeating digits followed by blocks of n repeating digits, the denominator is $(10^n - 1)(10^m)$

SUGGESTIONS FOR FURTHER STUDY:

- Use algebra to find the fractional representation of the decimal given in Conclusions question 1 above.
- Use algebra to find the fractional representation of the decimal given in Conclusions question 3 above.
- Use algebra to prove that $0.99999... = 1$.
- Try to discover a rule for finding the numerator of the decimal representation of a repeating decimal.

TEACHER'S GUIDE NOMOGRAPH

GOAL: The student will develop an understanding of addition and subtraction of integers.

STUDENT OBJECTIVE:

✓ To construct an instrument for calculating sums and differences of integers by examining the relationships of the positions of the integers on the number line.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this investigation are basic knowledge about integers, the number line, and the operations of addition and subtraction.

In this investigation each student should construct a nomograph, but cooperative learning groups may be used when students are working with the device to find sums and differences. If, for example, students are working in groups of three, one student could find the solution to a problem using the nomograph, a second could use the calculator, and the third could find the answer by applying previously learned computational rules. They may then compare their answers and discuss the processes that were employed.

If facilities are available to laminate the nomographs, students will be able to mark on them using water-based markers, then clean and reuse them.

VOCABULARY: integers, addends, sum, minuend, subtrahend, difference, number line, scale, intersect

SUGGESTED PATH FOR REMEDIATION: A chip trading model can also be used to assist students who are having difficulty with addition and subtraction of integers. In this model two color chips are used. These chips are usually yellow on one side and red on the other. The yellow side can be assigned a value of positive one and the red side a value of negative one. Hence, a yellow chip and red chip lying side by side has a value of zero. Since zero is the additive identity, it may be added to any number without changing the value of that number. To model the number 7, 7 yellow chips can be used, or 9 yellows and 2 reds, etc.

To model addition with red chips and yellow chips, model the two addends, place them together, and remove all the zeros you can. The resulting pile of chips represents the sum. For example, – 5 + 3 could be modeled with 5 red chips and 3 yellow chips. When you combine the chips for these addends, you can make three zeros, leaving 2 red chips. Thus, the sum is – 2.

To model subtraction, model the minuend and then remove chips equal in value to the subtrahend. You may need to place zeros (pairs of red chips and yellow chips) on the work area before the subtrahend can be removed.

For example:
– 3 – (– 5) could be modeled by placing 3 red chips in the work area. To subtract – 5, you must remove 5 red chips, but there are not 5 red chips on the work area. You must place 2 zeros (2 yellow and red pairs) on the work area. The starting value is still – 3. Now remove – 5 (5 reds). There remain only two yellow chips in the work area. Therefore, the difference is 2.

ADDITIONAL RESOURCES: The National Council of Teachers of Mathematics 1978 Yearbook, *Developing Computational Skills.*

NOMOGRAPH

INTRODUCTION: Addition and subtraction of integers can be done using the relationship of integers' locations on the number line.

PURPOSE:

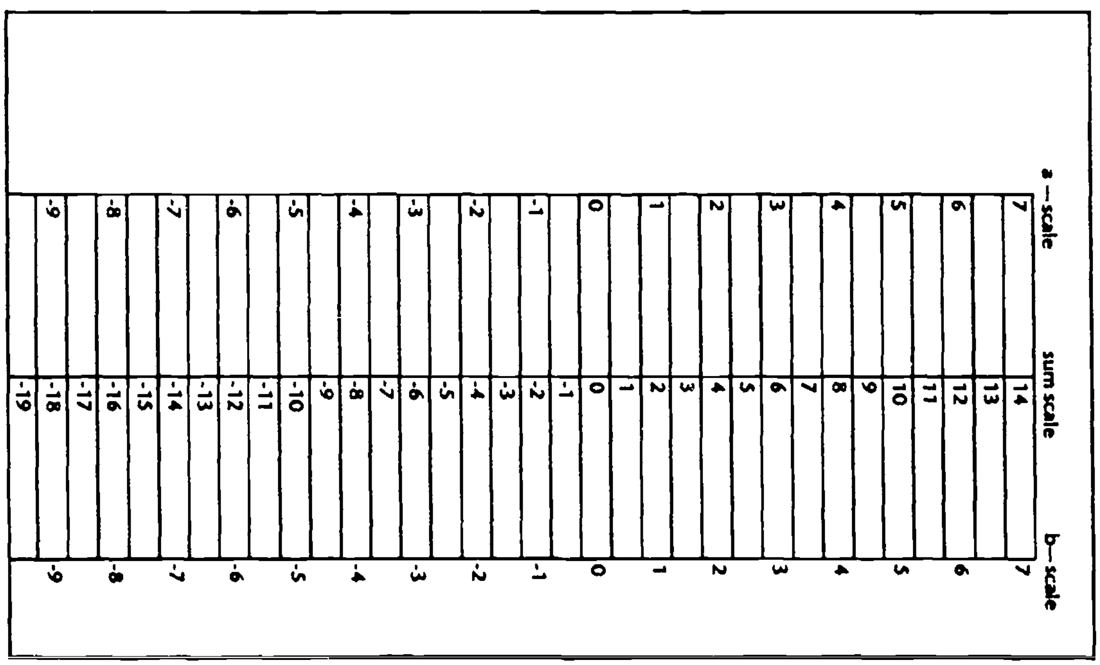
✓ Can you construct an instrument for calculating sums and differences of integers by examining the relationships of positions of the integers on the number line?

MATERIALS:

graph paper
straight edge

PROCEDURES:

1. On the graph paper draw three horizontal, parallel lines with one line midway between the other two. Label the line on the top as the a-scale, the line on the bottom, the b-scale, and the centerline, the sum-scale. See illustration 17–1.
2. Mark the coordinates on the lines as follows: the a-scale and the b-scale should be marked off in identical units, but the sum scale should be marked off in units that are one-half the size of the units on the other two. The zero positions on all three lines should be aligned. This is your nomograph.



17 - 1

3. If facilities are available to laminate the piece of graph paper, this will enable you to draw on the graph using erasable water-based markers and reuse the nomograph many times.
4. To locate the sum of two addends, locate the first addend on the a-scale, and locate the second addend on the b-scale. Use the straight edge to draw a line segment connecting these two points. This line segment will intersect the sum-scale. The coordinate of this point of intersection will be the sum of the two addends.
5. To find a difference locate the minuend on the sum-scale and the subtrahend on the a-scale. Use a straight edge to draw a line segment connecting these two points and intersecting the b-scale. The coordinate of the point where this line segment intersects the b-scale is the difference.

OBSERVATIONS:

1. Use the nomograph to do several addition problems. Record the addends you use and the sums on the chart below.

First Addend	Second Addend	Sum

2. Use the nomograph to do several subtraction problems. Record the minuend, subtrahend, and difference in the chart below.

Minuend	Subtrahend	Difference

CONCLUSIONS:

1. Write a paragraph describing reasons why the nomograph works for addition.

Answers vary. Should include an explanation of why sum scale is ½ of a-scale, and a-scale = b-scale

2. Write a paragraph describing reasons why the nomograph works for subtraction.

$a + b = c \rightarrow c - a = b$ and $c - b = a$

SUGGESTIONS FOR FURTHER STUDY:

- A nomograph for multiplication can be constructed using the formula $\log ab = \log a + \log b$. On the a-scale and b-scale use a standard logarithmic scale. On the product scale (center) use a double logarithmic scale. When using a nomograph for multiplication one must estimate values that are located between the marked points.

TEACHER'S GUIDE
FOOTBALL ARITHMETIC WITH INTEGERS

GOAL: To help students develop a better understanding of integer arithmetic.

STUDENT OBJECTIVE:

✓ To explore operations with integers using a football field model.

GUIDE TO THE INVESTIGATIONS: The prerequisite for this activity is previous instruction on using integers to represent values that are above and below some origin (starting point), called zero.

Many students learn to do integer arithmetic very quickly, while others struggle. We present one way to reinforce the concepts related to integer arithmetic. There are, of course, others, some of which will work better for some students, some not so well. Many teachers find it necessary to review integer arithmetic many times, so having a variety of instructional strategies is beneficial.

During this investigation students should work in cooperative groups, debating and discussing the best ways to model problem situations as well as developing strategies for finding results without the aid of the model.

VOCABULARY: integer, positive, negative, net result

SUGGESTED PATH FOR REMEDIATION: This investigation is designed to help students who need remedial work with integer arithmetic. If this investigation does not produce a successful result with some students, you may want to employ other strategies, such as activity 17 "Nomograph", in this volume of *GEM*.

ADDITIONAL RESOURCES: *Algebra for Everyone,* edited by Edgar Edwards, Jr., (National Council of Teachers of Mathematics) is an excellent resource for teachers as they strive to provide meaningful instruction for all students.

FOOTBALL ARITHMETIC WITH INTEGERS

INTRODUCTION: In this investigation you use the model of a football field to explore operations with integers. The line of scrimmage at the beginning of a down will always correspond to zero (the origin); lost yardage will correspond to negative integers; gained yardage, to positive integers.

PURPOSES:

- ✓ Can a football field be used to help develop an understanding of integer arithmetic?
- ✓ What are the rules of integer arithmetic?

MATERIALS:

butcher paper
ruler
yardstick
white spray paint
permanent markers

PROCEDURES:

1. Spray paint the yardstick and use a permanent marker to mark the center of the stick with the number 0. To the right of zero mark off equal units each of a half-inch, and label these 1, 2, 3, 4,.... To the left of zero, do likewise, but label these – 1, – 2, – 3,.... This will be your number line.
2. On a piece of butcher paper about six feet long, draw a football field. Let each half-inch represent one yard. Thus, from goal line to goal line the total distance of 100 yards is represented by fifty inches. Complete the football field by drawing side-lines and end-zones to scale. (The width of a standard football field is 160 feet, and end-zones are each 10 yards long.)
3. To model a problem, place the number line on the field so that the point marked zero is lying on the line of scrimmage. To model a loss, move in the negative direction on the number line; to model a gain, move in the positive direction. You will need some kind of small marker to move up and down the number line to represent the position of the football.

OBSERVATIONS:

1. Place the number line on the field so that the beginning line of scrimmage is the 30 yard line of the home team. The opposing team has the ball. Suppose that they gain five yards on the first play and lose three on the second. What was the net gain or loss on the two plays? Where is the marker on the number line?

   +2

   What is $5 + (-3)$?

   +2

2. Place the number line so that the zero is on the 45 yard line of the opposing team. This represents the line where the last first down was made by the home team. Suppose that the home team was twelve yards behind the line before the last play and that they will be three yards ahead of it after this play. What is the net change in their position over the two plays?

   + 15

   What is $3 - (-12)$? Or, what number can be added to -12 to get 3? Or, $-12 +$ what $= 3$?

   + 15

3. Place the number line so that the zero is on the 20 yard line of the opposing team. This represents the line of scrimmage. Suppose that the home team has the ball and loses three yards on each of the next two plays. What is the net result of the two plays?

   – 6

   What is $2(-3)$?

   – 6

4. Place the number line so that the zero is on the 50 yard line. This represents the line where the last first down was marked. Suppose that the home team has the ball and loses three yards on each of the next two plays. How well will they have to do on the next play to get back to the location of the last first down?

   + 6

   What is $(-2)(-3)$? Or, $(2)(-3) +$ what $= 0$. Or, what is the opposite of 2 times negative 3?

   + 6

5. Place the number line so that the zero is on the 60 yard line of the home team. This will represent the line at which the last first down was marked. After three plays the home team had lost 12 yards from the point at which they had taken possession. If they had exactly the same change in yardage in each of the three plays, what was the yardage for each of the three plays?

   –4

   What is $\frac{-12}{3}$?

   –4

CONCLUSIONS:

1. Use the problems in the Observation section as models and create twelve other problems. Make sure that you use a variety of operations. Write these problems. Then model them.
2. Write a paragraph about what you have learned about integer arithmetic.

   Answers will vary. Things to look for include the adding of positive numbers to get still larger numbers and the effect of "double reverses," i.e. –(–2) is +2.

SUGGESTIONS FOR FURTHER STUDY:

- It is very difficult to write a model for a positive integer divided by a negative integer. This may be a challenge that some students wish to undertake.
- Investigate other models for integer arithmetic, such as using colored chips.

TEACHER'S GUIDE
POPCORN

GOAL: To help students understand the concept of statistical variation.

STUDENT OBJECTIVES:

- ✓ To explore the importance of variability in measures of central tendency.
- ✓ To use standard deviation as a measure of central tendency.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity include the concept of mass, ability to measure mass, the ability to compute percent of loss, basic knowledge of measures of central tendency such as mean, median, and mode, knowledge of the range as crude measure of variability, and beginning knowledge of standard deviation as a measure of variability.

The experiment described in the Procedures section should be conducted by students in small cooperative groups, with each group having access to the necessary equipment. As groups conduct the experiment, they should record their findings on the chart provided and answer the questions in the Observation and Conclusion sections.

VOCABULARY: mass, gram, measure of central tendency, mean, median, mode, measures of variability, range, standard deviation, percent of loss

SUGGESTED PATH FOR REMEDIATION: Students may need some review in order to compute percent of mass loss in the chart. The following may be helpful in this review process:

Mass loss is what percent of mass before popping?

Mass loss $= \frac{r}{100}$ times (mass before popping)

Percent of loss $= \frac{\text{mass loss}}{\text{mass before popping}}$ times 100

Some students may also need to review development of formulas for mean and standard deviation.

ADDITIONAL RESOURCES: *The Quantitative Literacy Series: Exploring Data*, by James M. Landwehr and Ann E. Watkins (Dale Seymour Publications) is an excellent source of activities related to statistics.

POPCORN

INTRODUCTION: As heat is applied to a kernel of popcorn, the water in the kernel evaporates, causing the kernel to pop. This activity is designed as an exploration of the variability of the mass of kernels of popcorn before popping, the variability of the mass of kernels of popcorn after popping, and the variability of the amount of water in kernels of popcorn before popping.

PURPOSES:

✓ How does the mass of kernels of popcorn vary before popping?

✓ How does the mass of kernels of popcorn vary after popping?

✓ How does the amount of water in kernels of popcorn vary before popping?

MATERIALS:

popcorn
Erlenmeyer flask
Bunsen burner or other heat source
test tube clamp
laboratory scale which measures to the thousandth of a gram

PROCEDURES:

1. Select ten kernels of popcorn at random and determine the mass of each to the nearest thousandth of a gram. Record these in the table on the next page.
2. One at a time, place a kernel in an Erlenmeyer flask and shake it over a flame until it pops. Use a test tube clamp to hold the flask. Determine the mass of each kernel after popping. Record these in the table on the following page.
3. Complete the other columns in the table.

Kernel Number	Mass Before Popping	Mass After Popping	Mass Loss	Percent Mass Loss
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

4. Calculate the mean, range, and standard deviation for each of the columns in the table.

Answers depend on measures.

OBSERVATIONS:

1. Compare the ranges for the columns.

Ranges for mass before popping, mass after popping, and mass loss should be relatively varied. Range for percent mass loss should be smaller.

2. Compare the standard deviations for the columns.

Standard deviation for percent of mass loss should be smaller, indicating less variability.

CONCLUSIONS:

1. What are some possible explanations for the differences in the standard deviations for the columns?

   Population size—accuracy and consistency increase with greater numbers.

   Faulty sampling technique—need to be random.

2. Are the findings from comparing the ranges for the columns consistent with the findings for the comparison of the standard deviations for the columns?

   They should be fairly close.

SUGGESTIONS FOR FURTHER STUDY:

- What percent of the data points for each column falls within one standard deviation of the mean?
- What percent of the data points for each column falls within two standard deviations of the mean?
- Research normal distributions (the normal curve). Would you say the masses of the kernels of popcorn before popping are normally distributed? Would you say that the masses of the kernels of popcorn after popping are normally distributed?

CONNECTIONS

TEACHER'S GUIDE
THE MATHEMATICS OF MEDICINE

GOALS: To develop students' understanding of exponential functions.

STUDENT OBJECTIVES:

✓ To create a sequence representing the amount of medication in a patient's system as time changes, given the amount of medication taken and the rate at which body fluid is excreted and replaced.

✓ To create a sequence representing the amount of medication in the system as time changes, given the dosage of medication, the interval at which the dosage is repeated, and the rate at which body fluid is excreted and replaced.

✓ To determine the desired dosage of medication to stabilize the level of medication in the body at the desired amount, given the rate at which body fluid is excreted and replaced.

GUIDE TO INVESTIGATION: The prerequisite skills for this activity include knowledge of exponential notation, recursive notation, function notation, the formula for the sum of a geometric series, and the ability to use a calculator with recursion capability.

The demonstration described in the Procedures section of this activity can be conducted in a large or small group setting. This demonstration requires a clear glass container which will hold ½ gallon of water, red food coloring, a measuring cup, and a graduated cylinder. Following the demonstration, each student or cooperative group should complete the Observations and Conclusions sections. We recommend a programmable graphics calculator such as the Texas Instruments TI-81 for this activity.

VOCABULARY: sequence, series, recursive notation, function notation, limit of a sequence, sum of a geometric series, natural number

SUGGESTED PATH FOR REMEDIATION: One of the areas that may cause students some problems with this activity is the fact that loss of ¼ of a measurable thing is equivalent to retaining ¾ of that same thing. Some of the notation may also need to be explained to less experienced students.

Many students will probably need assistance in writing the function which gives the amount of medication in the system after time t has elapsed when a dose of medication is taken every four hours. Making a chart like the one below and reviewing the formula for the sum of a geometric series may be helpful.

Time Period	Amount of Medication
0	16
1	$.75(16) + 16$
2	$.75[.75(16) + 16] + 16 = .75^2(16) + .75(16) + 16$
3	$.75[.75^2(16) + .75(16) + 16] + 16 = .75^3(16) + .75^2(16) + .75(16) + 16$
n	$75^n(16) + .75^{n-1}(16) + \ldots + .75^2(16) + .75(16) + 16$

This is a geometric series with a ratio of .75, and the formula for finding the sum of this series is given in most algebra texts as

$$S_n + \frac{a(1 - r^{n+1})}{1 - r}$$

where *a* is the first term, *r* is the common ratio, and *n* is a natural number.

ADDITIONAL RESOURCES: This and related problems are discussed in an article "Drugs and Pollution in the Algebra Class" in the February, 1992, *Mathematics Teacher* (Volume 85, Number 2).

THE MATHEMATICS OF MEDICINE

INTRODUCTION: Medicines disperse through the body fluids. As we void and replace body fluid, the amount of medication in our system decreases unless we take more. By repeatedly taking doses of medication, we can stabilize the level of medication in our system at the amount desired to achieve maximum benefit.

PURPOSES:

- ✓ How long will it take for a dose of medication to be voided from the body?
- ✓ If you take medication at regular intervals, how does this effect the amount of medication in the body?
- ✓ If you desire to reach and maintain a certain level of medication in the body, how do you determine the proper dosage?

MATERIALS:

clear container (at least $\frac{1}{2}$ gallon)
water
red food coloring
metric measuring cup
graduated cylinder
calculator (a programmable graphics calculator is best)

PROCEDURES:

1. Pour 4 cups of water in the container. This represents the body and its fluid.
2. Use the graduated cylinder to remove 16 mL of water and replace it with 16 mL of red food coloring. Stir to mix. The red food coloring represents medication in the body.
3. Assume that every four hours one quarter of the body fluid is lost and replaced. To model this process, use the measuring cup and take out one cup of fluid and replace it with one cup of water. Some medicine (How much?) is lost and not replaced.
4. Model the passing of yet another four hours by removing another one cup of colored fluid and replacing it with a cup of water. The color of the water is representative of the amount of medication in the body. More has been lost. (How much?)
5. Continue modeling the passage of time in units of four hours. Each period results in the removal of one cup of fluid and the addition of one cup of water.

OBSERVATIONS:

1. What happens to the color of the water during this experiment? Why does it happen?

   Color becomes lighter, because there is less food coloring in the water.

2. Calculate the amount of medication in the body after the first four hours, eight hours, twelve hours, etc. Record these in the table.

Time Period	Amount (mL)
Beginning (0)	16
1 period	12
2	9
3	6.75
4	5.0625
5	3.796875
6	2.84765625
7	2.1357421875
8	1.60180664063
9	1.20135498047
10	.901016236352

3. Suppose that you are going to have to take a drug test, and the active ingredients in the medication you have taken will show up in the test if you have 1 mL or more of the drug remaining in your system. Calculate how long you will have to wait before you take the drug test if you do not want this medication to be detected. Use the calculator to solve this problem. (Using the Texas Instruments TI-81 Graphics Calculator, you can enter 16 to represent the 16 mL of medication taken. Every four hours you are losing a quarter of your body fluid, so you are losing a quarter of the medication which remains in your system. Enter *ANS – .25ANS* or *.75ANS* in your calculator. Each time you subsequently press Enter, the calculator will compute the amount of medication in your system at the end of another four hours. Count the number of times you must press Enter before the amount of medication is less that 1 mL.)

   10 time periods x 4 hrs/each = 40 hrs.

4. If the test could detect the medication, how many hours would you have to wait to take the test, if there is 0.1 mL or more in your body?

18 time periods x 4 hrs / period = 72 hrs.

5. Drug tests can actually detect extremely small amounts of drug residue. How long would you have to wait to take the test if the test could detect the medication, if there is 0.01 mL or more in your body?

26 time periods x 4hrs / period = 104 hrs.

6. Now suppose that in the example above you took 16 mL of medicine every four hours and that you lost and replaced one-quarter of your body fluid every four hours. Use the calculator to complete the table provided. (On the TI-81 enter 16 and then type the expression .75ANS + 16. This expression represents the three-quarters of the medication you retain after losing a quarter of body fluid plus the new 16 mL dose of medication you take every four hours.)

Time Period	Amount (mL)
Beginning (0)	16
1 period	28
2	37
3	43.75
4	48.8125
5	52.609375
6	55.45703125
7	57.5927734375
8	59.1945800781
9	60.3959350586
10	61.2969512939

CONCLUSIONS:

1. If you take only one dose of medication, in theory how long will it be before all the medication is gone from your body?

In theory, it is never all gone. One could specify a time past which only a given fraction, such as one millionth, would remain.

2. If you take a dose of medication every four hours, does the amount of medication in your body increase without limit?

   No.

3. If you take a dose of medication every four hours, does the amount of medication in your body ever stabilize?

   Yes, at 64 mL

4. Write a recursive definition for the sequence a_n which represents the amount of medication in your body after n time periods if you only take one dose of medication.

   $A_n = A_{n-1}(.75)$

5. What is the limit of this sequence as n tends to infinity? $\lim_{n \to \infty} a_n = ?$

   0

6. Write a function $f(n)$ that gives the amount of medication in your body after n four hour time periods if you take only one dose of medication. (If you have a graphics calculator, graph this function.)

   $f(n) = .75^n (16)$

7. What is the limit of this function as n tends to infinity? $\lim_{n \to \infty} f(n) = ?$

   0

8. Write a recursive definition for the sequence b_n which represents the amount of medication in your body after n time periods if you take a new dose of medication every four hours.

   $b_n = 16 + .75b_{n-1}$

9. What is the limit of the sequence as n tends to infinity?

64 mL

10. Write a function $g(n)$ that gives the amount of medication in your body after n four hour time periods if you take a dose of medication every four hours? (Graph this function if you have a graphics calculator.)

$g(n) = 64\,(1 - (3/4)^n)$

11. What is the limit of this function as n tends to infinity? $\lim_{n \to \infty} g(n) = ?$

64

SUGGESTIONS FOR FURTHER STUDY:

- If you take a single dose of 20 mL of medication, and if you lose and replace one-eighth of your body fluid every three hours, how long will it take for the level of medication in your system to go below 0.5 mL?
- Use sequence and function notation to represent this situation.
- If you take 20 mL of medication every three hours and you lose and replace one-eighth of your body fluid every three hours, write a sequence c_n and a function $h(n)$ to represent the situation.
- Calculate $\lim_{n \to \infty} c_n = ?$ and $\lim_{n \to \infty} h(n) = ?$.
- Suppose you lose and replace one-forth of your body fluid every four hours. What dosage of medication should be taken every four hours to reach and maintain a level of 48 mL of medication in your system?
- Suppose that you lose and replace one-eighth of your body fluid every three hours. What dosage of medication should be taken every three hours to reach and maintain a level of 48 mL in your system?

TEACHER'S GUIDE
HOW MANY BEANS ARE THERE?

GOAL: Students will develop an understand of sampling techniques, ratio, and proportion.

STUDENT OBJECTIVES:

- ✓ To devise a sampling technique providing data necessary to estimate the size of a population.
- ✓ To use ratio and proportion to derive an equation which will yield an estimate of population size based on the capture–recapture sampling technique.
- ✓ To solve ratio and proportion equations for an unknown.
- ✓ To find the percent of error in population estimation when the actual population is also known.

GUIDE TO THE INVESTIGATIONS: The prerequisite concepts for this activity are ratio, proportion, and percent.

This activity can be done as a whole class activity, if only one jar is used, or a cooperative group activity when each group is given a jar of beans. Using cooperative groups for this activity allows for more participation by individual students. The group approach also provides the class with an opportunity to compare results from the various groups after the activity has been completed. This may provide an opportunity to talk about sampling error and related topics.

Students may wonder what to do if a part of a bean is drawn in one of the samples. There are several options: each part of a bean can be treated as a whole bean, the parts of beans can be ignored if they are drawn, or any part of a bean that appears to be larger than a half bean can be counted, and any part of a bean less than a half a bean can be ignored. Students may make a group decision.

VOCABULARY: sampling techniques, capture–recapture, population, ratio, proportion, percent

SUGGESTED PATH FOR REMEDIATION: Some students may need additional instruction in solving ratio and proportion equations. There are several techniques. Many students find cross multiplication easy to remember and use, but they may not understand the significance of what they are doing. More time consuming, but perhaps enlightening, is to guess a value to use to replace the variable in the proportion and then use a calculator to convert the two ratios to decimals to compare the results. After the initial comparison the student makes a second guess adjusting for the results from the first guess. This guess–test–revise procedure is

continued until students get a result which makes the two ratios approximately the same. This approach gives students an opportunity to work with the guess-test-revise problem solving strategy, but the time that this procedure may require motivates them to seek other processes for finding solutions that are less time consuming.

ADDITIONAL RESOURCES: There are additional examples of how data collection and analysis can be integrated in the standard high school mathematics curriculum in *Data Analysis and Statistics Across the Curriculum: Addenda Series, Grades 9–12*, by Gail Burrill, John C. Burrill, Pamela Coffield, Gretchen Davis, Jan de Lange, Diann Resnick, and Murray Siegel (National Council of Teachers of Mathematics).

HOW MANY BEANS ARE THERE?

INTRODUCTION: Sometimes it is not practical or possible to count to find the number of elements in a population. Finding the number of fish in a breeding pond or the number of deer in a game preserve are two such examples. This investigation introduces a sampling technique known as capture–recapture, which can be used in situations when it is not possible to count the population directly.

PURPOSES:

✓ Can a sampling technique be devised using ratio and proportion to estimate the size of a population when it is not practical to count?

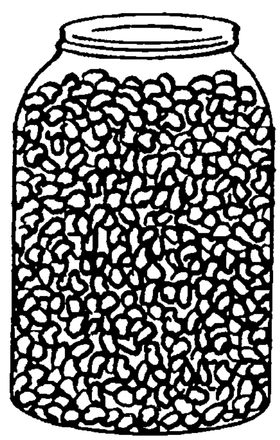
✓ How accurate is this technique?

MATERIALS:

gallon jar and lid
several pounds of dried beans
indelible, nontoxic laundry marker

PROCEDURES:

1. Fill the jar with the dried beans so that the beans are within about two inches of the top.
2. Run a contest in your class allowing each of your fellow students to guess the number of beans in the jar. Record the guesses of your classmates on the following table.



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Number of Beans in the Jar

Name	Guess	Name	Guess

3. Draw out a reasonable number of the beans from the jar, and mark them with the laundry marker. Lay these beans out to dry before placing them back in the jar. Be sure that you count and record in the Observations section the exact number of beans that were marked.

4. Place the marked beans back in the jar, place the lid on the jar, and shake the jar well to mix the marked and unmarked beans as thoroughly as possible. You may even want to put all the beans in a large sack to mix them, replacing them in the jar after you are satisfied that they are randomly shaken up.

5. Open the jar and draw out a cup or so of beans. Count the total number of beans that you have drawn out and record this in the Observations section.
6. Count the number of marked beans in this sample, and record this number in the Observations section.
7. Place all the beans back in the jar, and shake well again. Repeat steps 5 and 6 five times, recording the results in the Observations section.
8. After you have completed the Observations and questions 1–11 in the Conclusions section, divide the beans among the class and count them all.

OBSERVATIONS:

1. What is the average (mean) of the guesses of your classmates?

Answers vary. $\dfrac{\textit{Sum guesses}}{\textit{\# of guesses}}$

2. Exactly how many beans did you mark?

Answers vary.

3. Record the results of your sampling in the table below.

Sample Trial #	# Beans	# Marked Beans
1		
2		
3		
4		
5		

CONCLUSIONS:

1. Write a proportion which can be used to estimate the number of beans in the jar, using the data from the first trial.

Answers will vary. $\dfrac{\textit{\# marked in trial 1}}{\textit{\# in trial 1}} = \dfrac{\textit{\# marked in jar}}{\textit{total } (x)}$

2. Solve the equation you derived in number 1.

$x = \dfrac{(\textit{\# marked in jar})(\textit{\# marked in trial 1})}{\textit{\# in trial 1}}$

3. Write a proportion that can be used to estimate the number of beans in the jar, using the data from the second trial.

Similar to # 1

4. Solve the equation.

Similar to # 2

5. Write a proportion that can be used to estimate the number of beans in the jar, using the data from the third trial.

Similar to # 1

6. Solve the equation.

Similar to # 2

7. Write a proportion that can be used to estimate the number of beans in the jar, using the data from the fourth trial.

Similar to # 1

8. Solve the equation.

Similar to # 2

9. Write a proportion that can be used to estimate the number of beans in the jar, using the data from the fifth trial.

Similar to # 1

10. Solve the equation.

Similar to # 2

11. What is the average (mean) of these five estimates?

Add estimates and divide by 5.

12. Compare the actual number of beans in the jar with your estimate and with the guesses made by your classmates. Find the percent of error for your estimate.

See formula below.

$$\text{Percent of error} = 100\,\frac{|\,\text{actual number} - \text{estimated number}\,|}{\text{actual number}}$$

SUGGESTIONS FOR FURTHER STUDY:

- Call or write your local Game and Fish Commission to inquire about the types of sampling techniques they use to estimate various wild life populations in your state.
- Call your local Audubon Society to ask how banding is used to estimate bird populations. The Society may be able to send a speaker to explain their work to the class.

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TEACHER'S GUIDE
POLYGONAL NUMBERS

GOAL: The student will develop an understanding of the connection between geometry and algebraic sequences and series.

STUDENT OBJECTIVES:

✓ To define polygonal number, the number corresponding to a given regular polygon with n sides.

✓ To discover the relationship between two or more different polygonal numbers of a given rank.

GUIDE TO THE INVESTIGATIONS: The prerequisites for this investigation are knowledge of regular polygons, sequences, and series. Students also need some experience with generalizing, especially in writing generalized formulas for sequences. They may work individually or in cooperative groups to complete this investigation. If cooperative groups are used, each group should produce the pictorial models on a large sheet of butcher paper as instructed in the Procedures section so that the models can be seen and studied by all the members of the group as they work on the Observations and Conclusions sections. Use of cooperative groups will allow all students to make a contribution during the construction of the models and to take part in finding the terms of the sequences by counting the points on these models. Group give and take may be very helpful in deriving the formulas for the sequences.

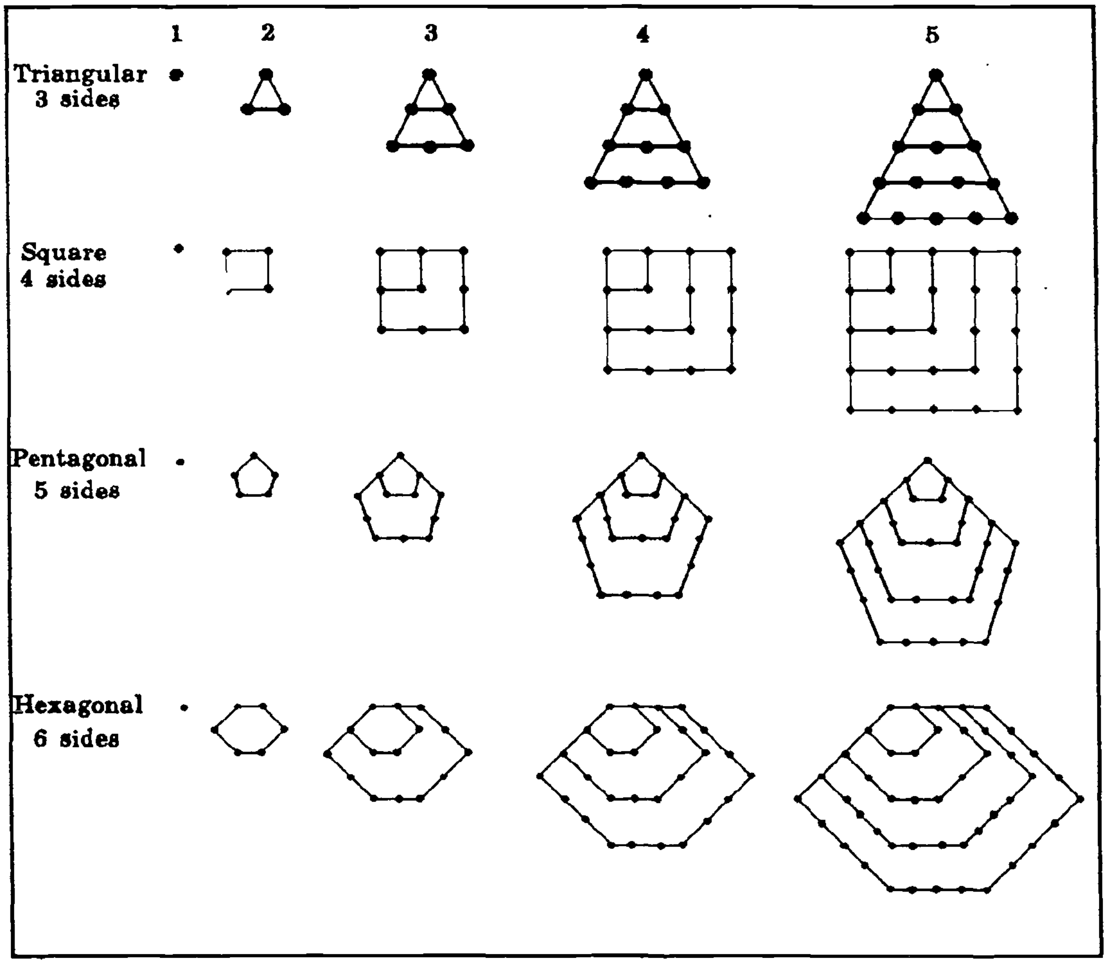
VOCABULARY: polygon, triangle, square, pentagon, hexagon, heptagon, octagon, nonagon, decagon, n-gon, series, sequence

SUGGESTED PATH FOR REMEDIATION: Students having trouble drawing the pictorial models required in this investigation may need to review the definition of regular polygons. They may need to be reminded that the measure of each angle of a regular polygon with n-sides is $\frac{(n-2)(180)}{n}$. If students are having trouble getting started drawing the models, suggest that they begin with a triangle with sides all measuring 1 inch and angles all measuring 60°. The next triangle will have sides 2 inches and 60° angles. In the second triangle mark off one inch segments on each of the three sides, connecting the midpoints of two of the sides. The third triangle will have sides 3 inches and 60° angles. In this triangle, mark off one inch segments on the three sides, connecting the one inch marks on two of the sides. Continue this process to the ninth triangle, which will be 9 inches on all sides. This exercise may be of help.

ADDITIONAL RESOURCES: *Teaching Mathematics: A Sourcebook of Aids, Activities, and Strategies, Second Edition*, by Max A. Sobel and Evan M. Maletsky (Prentice Hall, 1988) includes some information on polygonal numbers.

POLYGONAL NUMBERS

INTRODUCTION: Numbers which can be related to geometric figures are called figurative or polygonal numbers. The table below shows the first five ranks of the first four figurative numbers.



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PURPOSES:

- ✓ Given the rank of any regular polygon, how can you find the number that corresponds to it?
- ✓ What is the formula for the r^{th} n-gonal (figure with n congruent sides) number?

MATERIALS:

butcher paper	ruler
protractor	markers

PROCEDURES:

1. Reproduce the table shown in the Introduction, expanding the table to include the 6th, 7th, 8th, and 9th ranks of the triangular, square, pentagonal, and hexagonal numbers. Do this reproduction on a large sheet of butcher paper, using the protractor, ruler, and markers.
2. Expand the table to show the first 9 ranks of the heptagonal (7-sided) numbers.
3. Expand the table to show the first 9 ranks of the octagonal (8-sided) numbers.
4. Expand the table to show the first 9 ranks of the nonagonal (9-sided) numbers.

OBSERVATIONS:

1. Fill in the chart below for the first five ranks of the triangular, square, pentagonal, hexagonal, and heptagonal numbers. Use the pictorial models in the Introduction to accomplish this task. Each blank should be filled with the number of points on the figure.

Figure	n	1	2	3	4	5
triangular	3	1	3	6	10	15
square	4	1	4	9	16	25
pentagonal	5	1	5	12	22	35
hexagonal	6	1	6	15	28	45

2. Expand the following sequence of sums for triangular numbers.

Rank	Sum	Total
1	1 =	1
2	1 + 2 =	3
3	1 + 2 + 3 =	6
4	1 + 2 + 3 + 4 =	10
5	1 + 2 + 3 + 4 + 5 =	15
6	1 + 2 + 3 + 4 + 5 + 6 =	21
7	1 + 2 + 3 + 4 + 5 + 6 + 7 =	28
8	1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 =	36
9	1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 =	45
r	1 + 2 + 3 + ... + r	$\frac{r(r+1)}{2}$

3. Expand the following sequence of sums for square numbers.

Rank	Sum	Total
1	1 =	1
2	1 + 3 =	4
3	1 + 3 + 5 =	9
4	1 + 3 + 5 + 7 =	16
5	1 + 3 + 5 + 7 + 9 =	25
6	*1 + 3 + 5 + 7 + 9 + 11 =*	*36*
7	*1 + 3 + 5 + 7 + 9 + 11 + 13 =*	*49*
8	*1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 =*	*64*
9	*1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 =*	*81*
r	1 + 2 + 3 + ... + (2r - 1)	r(r)

4. Create a sequence of sums for the pentagonal numbers.

Rank	Sum	Total
1	1 =	1
2	1 + 4 =	5
3	1 + 4 + 7 =	12
4	1 + 4 + 7 + 10 =	22
5	1 + 4 + 7 + 10 + 13 =	35
6	*1 + 4 + 7 + 10 + 13 + 16 =*	*51*
7	*1 + 4 + 7 + 10 + 13 + 16 + 19 =*	*70*
8	*1 + 4 + 7 + 10 + 13 + 16 + 19 + 22 =*	*92*
9	*1 + 4 + 7 + 10 + 13 +16 + 19 + 22 + 25 =*	*117*
r	1 + 2 + 3 + ... + (3r - 2)	$\frac{r(3r-1)}{2}$

5. Create a sequence of sums for the hexagonal numbers.

Rank	Sum	Total
1	1 =	1
2	1 + 5 =	6
3	1 + 5 + 9 =	15
4	1 + 5 + 9 + 13 =	28
5	1 + 5 + 9 + 13 + 17 =	45
6	1 + 5 + 9 + 13 + 17 + 21 =	66
7	1 + 5 + 9 + 13 + 17 + 21 + 25 =	91
8	1 + 5 + 9 + 13 + 17 + 21 + 25 + 29 =	120
9	1 + 5 + 9 + 13 + 17 + 21 + 25 + 29 + 33 =	153
r	1 + 5 + 9 + ... + (4r − 3)	$\frac{r(4r-2)}{2}$

6. Create a sequence of sums for the heptagonal numbers.

Rank	Sum	Total
1	1 =	1
2	1 + 6 =	7
3	1 + 6 + 11 =	18
4	1 + 6 + 11 + 16 =	34
5	1 + 6 + 11 + 16 + 21 =	55
6	1 + 6 + 11 + 16 + 21 + 26 =	81
7	1 + 6 + 11 + 16 + 21 + 26 + 31 =	112
8	1 + 6 + 11 + 16 + 21 + 26 + 31 + 36 =	148
9	1 + 6 + 11 + 16 + 21 + 26 + 31 + 36 + 41=	189
r	1 + 6 + 11 + ... + (5r − 4)	$\frac{r(5r-3)}{2}$

7. Create a sequence representing the r^{th} term of the n^{th} figurative number.

n		
3	$1 + 2 + 3 + \ldots r =$	$\frac{r(r+1)}{2}$
4	$1 + 3 + 5 + \ldots + (2r - 1) =$	$\frac{r[(2r-1)+1]}{2}$
5	$1 + 4 + 7 + \ldots + (3r - 2) =$	$\frac{r[(3r-2)+1]}{2}$
6	$1 + 5 + 9 + \ldots + (4r - 3) =$	$\frac{r[(4r-3)+1]}{2}$
7	$1 + 6 + 11 + \ldots + (5r - 4) =$	$\frac{r[(5r-4)+1]}{2}$
8	$1 + 7 + 13 + \ldots + (6r - 5) =$	$\frac{r[(6r-5)+1]}{2}$
9	$1 + 8 + 15 + \ldots + (7r - 6) =$	$\frac{r[(7r-6)+1]}{2}$
n	$1 + (n-1) + (2n-3) + (3n-5) \ldots + (n-2)r - (n-3) =$	$\frac{r[[(n-2)r-(n-3)]+1]}{2}$

CONCLUSIONS:

1. Use the formula for the r^{th} term of the n-gonal number to find the 5th decagonal (10-sided) number.

2. What is the sum of the r^{th} triangular number and the r^{th} square number?

$$\frac{r(r+1)}{2} + \frac{r[2r-1)+1]}{2} = \frac{r^2+r}{2} + r^2 = \frac{r^2+r+2r^2}{2} = \frac{3r^2+r}{2}$$

3. What is the sum of the r^{th} triangular number and the r^{th} pentagonal number?

$$\frac{r(r+1)}{2} + \frac{r[(3r-2)+1]}{2} = \frac{r^2+r+3r^2-2r+r}{2} = \frac{4r^2}{2} = 2r^2 = \frac{r(3r+1)}{2}$$

4. What is the sum of the r^{th} m-gonal (m-sided) number and the r^{th} n-gonal (n sided) number?

$$\frac{r[r(m-2)-m+4]}{2} + \frac{r[r(n-2)-n+4]}{2} = \frac{r^2m-2r^2-mr+4r+r^2n-2r^2-nr+4r}{2}$$

$$= \frac{r^2(m+n)-4r^2-r(m+n)+8r}{2} = \frac{r^2(m+n-4)-r(m+n-8)}{2}$$

SUGGESTIONS FOR FURTHER STUDY:

- Suppose the length of each line segment connecting the points on the figures above is one unit. Study the sequence of areas of each of the figurative numbers. For example, the triangular areas would be, 0, $\frac{\sqrt{3}}{4}$, $\sqrt{4}$,.... Can you discover formulas for these area sequences?
- There is a famous story about the mathematician Frederick Gauss finding the sum of 1 + 2 + 3 + 4 + 5 + ... + 100 very quickly when he was a young man. Investigate this bit of mathematics history. How does this story apply to the sums in this lesson?

TEACHER'S GUIDE
DIOPHANTINE EQUATIONS

GOAL: The student will develop an understanding of Diophantine equations and their applications.

STUDENT OBJECTIVES:

- ✓ To develop methods for finding solutions to Diophantine equations.
- ✓ To find applications of linear and nonlinear Diophantine equations.

GUIDE TO THE INVESTIGATIONS: The prerequisites for this activity are the ability to graph equations in two variables and a basic understanding of number theory. The concept of greatest common factor is also important. This investigation begins with students producing a graph of a Diophantine equation. It attempts to motivate them to seek a systematic method for finding the solutions to the equation and encourages them to do research about the method for solving these equations developed by Euler. It is important that secondary students begin to experience conducting library research on topics in mathematics. This will benefit them not only in their study of mathematics but also in other areas.

VOCABULARY: Diophantine equations, linear, nonlinear, rectangular coordinate system, x-axis, y-axis, solutions, integral solutions, greatest common factor

SUGGESTED PATH FOR REMEDIATION: Some students may need a review of how to graph straight lines from equations of the form $ax + by = c$. They may benefit from a review of slope-intercept form. In addition to teaching students methods for graphing equations, the use of a graphics calculator such as the Texas Instruments TI-81 or a computer with graphing software can help students develop graphing skills.

ADDITIONAL RESOURCES: *An Introduction to the Theory of Numbers*, by Ivan Niven and H. S. Zuckerman (John Wiley and Sons, 1980)has been the standard text for decades.

DIOPHANTINE EQUATIONS

INTRODUCTION: Diophantine equations are equations whose solutions must be integers. They are named to honor the Greek mathematician Diophantus, who wrote about them.

PURPOSES:

✓ Are there methods for solving Diophantine equations?

✓ What are some of the applications of linear and nonlinear Diophantine equations?

MATERIALS:

graph paper
resource books on number theory
computer with graphing software or graphics calculator (optional)

PROCEDURES:

1. The current postage rate for first class letters weighing up to one ounce is 29 cents. Postcards are 19 cents. What are all the possible combinations of one-ounce letters and postcards we can mail with $11.88? Let x be the number of letters and y the number of postcards. The answer to our question, then, is the solution to the equation $29x + 19y = 1188$.

   You can get a feel for the infinite number of solutions to this equation by graphing the line it represents. You may wonder, though, about the existence of positive, integral solutions.

2. Locate the point $(5\frac{1}{2}, 54\frac{5}{38})$ on the graph. This ordered pair satisfies the equation, but this solution makes no sense in terms of letters or postcards. You are looking for solutions to this equation in which both components of the ordered pair (x, y) are positive integers. This is a Diophantine equation.

OBSERVATIONS:

1. Does the equation have any positive integral solutions?

   Yes. ____

   ____

   ____

2. If positive integral solutions exist, how many are there?

x = 1, y = 61; x = 20, y = 32; x = 39, y = 3; x = 58, y = –26; x = 17, y = –55

In general: x = 1 + n(19), y = 61 –n(29)

Negative numbers don't make sense in this problem.

CONCLUSIONS:

1. Consider the following theorem from number theory:

   If a, b, and k are integers and the greatest common factor of a and b is also a factor of k, then there exists an infinite number of integral solutions for x and y such that $ax + by = k$.

   Apply this theorem to the postal problem.

   29x + 19y = 1188

   x = 1, y = 61

   gcf (29,19) = 1

   1 is a factor of 1188

2. Once you have determined that a Diophantine equation has infinitely many solutions, how do you find them? What techniques did you apply in the Observations section to find the positive solutions to this particular Diophantine equation?

   Trial and error using the g.c.f.

   Enter $\left(\frac{1188 - 29x}{19}\right)$ *in a calculator and evaluate for different integral values of x.*

3. The Euler Method, named after the mathematician Leonhard Euler, is a systematic method for finding solutions to Diophantine equations. Consult resources to find this method and apply it to the problem under investigation.

4. Find other applications of Diophantine equations. Find the solutions to the equations using the methods you have found in your research.

   If you have $95.00 in fives and tens, how many of each can you have?

   If ax + by = k gcf(a,b)/k then x = gcf(a,b) + nb

   y = k/gcf(a,b) – na

SUGGESTIONS FOR FURTHER STUDY:

- Find through research and/or develop methods for finding solutions to Diophantine equations of higher degree. For example: What are the integral solutions to $x^2 + y^2 = z^2$? The solutions to this Diophantine equation are Pythagorean Triples.
- Another interesting equation to consider is $x^3 + y^3 = xy$.

TEACHER'S GUIDE
CARNIVAL COIN TOSS

GOAL: To help students develop a connection between geometry and probability.

STUDENT OBJECTIVES:

✓ To find the theoretical probability of winning a carnival coin toss game using an area model.

✓ To compare the theoretical probability with the empirical probability.

GUIDE TO THE INVESTIGATIONS: Here is how a coin toss at a carnival works. The player tosses a coin onto an area marked off in squares. If the coin lands so that it does not touch a line, then the player wins. Otherwise, the player loses his money. To play this game, you will need to have a very large sheet of poster board, a ruler or yardstick, a marker, and a penny, nickel, dime, quarter, and, if the bank has some, a half dollar, and a silver dollar. The game may be played by the students individually, but it will probably be more fun—and beneficial—if you arrange to have them do it in groups. If computers are available, you may extend the game by computer simulation to produce experimental probabilities.

Students must follow the directions in the Procedures section. Each group will need sufficient space in which to construct a game board, conduct the activity, and collect experimental data.

Discuss the results with the students, and have the groups compare their various outcomes. Pool all the experimental data, and compare these with what the theory predicts. What factors might have an effect on the outcome— the size of the squares, the size and weight of the coin, the distance of the players from the game board, etc.

VOCABULARY: empirical (experimental) probability, theoretical probability, ratio

SUGGESTED PATH FOR REMEDIATION: This exercise is relatively fun and straightforward. Some remedial instruction about fractions and ratios may be in order. It is also important that students realize that a probability is a number between 0 and 1, even when it is expressed as a percentage. Weather forecasters use probability, so you may want to talk about what they mean when they say, "The probability of rain tomorrow is 30%."

ADDITIONAL RESOURCES: The National Council of Teachers of Mathematics publishes a set of materials and software entitled *Geometric Probability* (1988), which was developed by the Department of Mathematics and Computer Science of the North Carolina School of Science and Mathematics. This package includes software which simulates the coin toss.

CARNIVAL COIN TOSS

INTRODUCTION: Have you ever seen a coin toss game at a county fair or carnival? There is always a large area (the board) which is marked into squares and onto which you toss a coin from behind a barricade or fence. If the coin lands in such a way that it does not touch any of the lines and is entirely inside a square, you win a stuffed animal. If not, you lose your money. The question, "What is the probability you will win?" is one which requires both probability theory and geometry to answer.

PURPOSES:

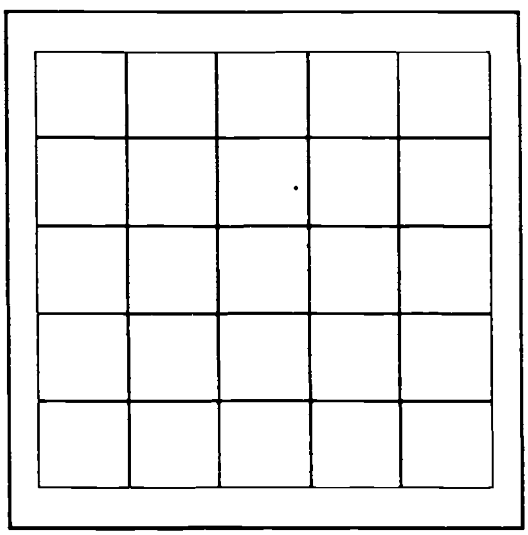
- ✓ What is the empirical probability that you can win at the Carnival Coin Toss game?
- ✓ What is the theoretical probability that you can win at the Carnival Coin Toss game?
- ✓ How do these compare?

MATERIALS:

poster board
ruler or yardstick
marker
one of each denomination of coins in circulation

PROCEDURES:

1. Use a ruler and marker to draw a 10–inch square on a piece of posterboard. Divide this square into 25 2-inch squares. This will be your game board.
2. Fix a distance from which to toss a penny at the game board. If it lands entirely inside a square, record the toss as a win; if not, a loss. Toss it fifty times, recording the outcome. There is a table which you may use in the Observations section.
3. Repeat the experiment with each of the other coins you were able to obtain: a nickel, a dime, a quarter, a half-dollar, and a dollar. Record the outcomes, as above.



24 · 1 Divide this 10-inch square into 25 2-inch squares.

4. Choose one of the coins and repeat the experiment from two distances other than the one you used above, one greater, and one lesser. Record the results.

OBSERVATIONS:

1. Complete the table below.

Coin	Wins	Losses
penny	*Answers vary.*	
nickel		
dime		
quarter		
half-dollar		
dollar		

2. What is the probability that you will win when you toss a penny? A nickel? A dime? Etc.

penny ≈ .44

nickel ≈ .35

dime ≈ .43

quarter ≈ .25

half-dollar ≈ .17

3. What coin did you choose for the distance experiment? Record the identity of the coin and your results here.

Answers vary.

Distance	Wins	Losses
greater	*Answers vary.*	
lesser		

4. What two factors seem to have the greatest effect on the experimental probability?

Size of the coin and distance from the game board

5. What is the geometric shape of the coin?

Circle

6. What is the geometric shape of each section of the game board?

Square

7. In order for you to win a toss with a given coin, how far must its center land from the edge of a square?

No farther than the measure of its radius.

CONCLUSIONS:

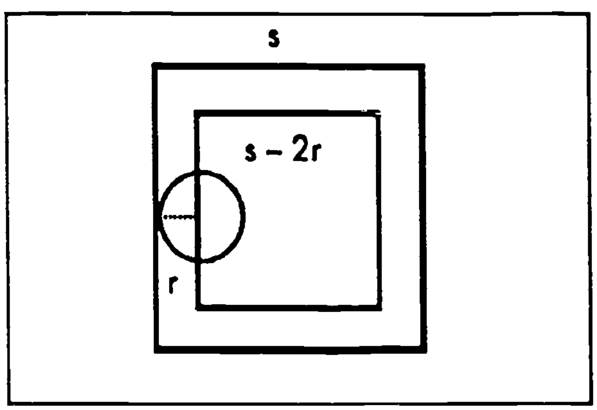
1. Refer to the experimental data you gathered, and give the experimental probability that you will win when you toss a penny, a nickel, a dime, and the other coins which were available to you to toss.

Answers will vary.

See # 2 in Observations.

2. Refer to the illustration 24-2. If the center of the coin lands within the center area, you will win. *r* stands for the radius of the coin, and *s* the length of a side of the square.

What is the area of the entire square?



24 - 2

s^2

4. What is the area of the smaller square, which will produce a win?

$(s - 2r)^2$

5. What is the ratio of the latter to the former? This is the theoretical probability that you will win.

$\left(\frac{s - 2r}{r}\right)^2$

6. Take the necessary measurements and compute the probability that you will win with each of the coins which was available to you for tossing.

Answers will vary.

See # 2 in Observations.

7. How do the empirical (experimental) results compare with the theoretical results for each coin you were able to use?

If the measurements were accurate and the experimental data sufficiently large they should be close.

SUGGESTIONS FOR FURTHER STUDY:

- Suppose that you are the owner of the coin toss concession and wish to make as much money as you can. What coin would you require to be used, and what size would you make the squares?
- Suppose that you are attending the fair. How would you make a decision about whether or not to take part in the coin toss?

TEACHER'S GUIDE
MASS –VOLUME CORRELATION

GOAL: To develop students' understanding of linear equations in slope-intercept form.

STUDENT OBJECTIVE:

- ✓ To generate a linear graph of the mass of a liquid versus its volume, the slope of which is the density of the liquid, and the y-intercept of which is the mass of the container holding the liquid.

GUIDE TO THE INVESTIGATION: The prerequisites for this activity include the abilities to use a scale to measure mass, to use a graduated cylinder to measure the volume, to plot points on a rectangular coordinate system, and to write the equation of a line in slope-intercept form when given two points on the line.

We recommend that students work in cooperative groups while carrying out this activity. Each group should be given a different liquid so that the densities will vary. The liquids might be artificially dyed different colors with safe food coloring. One group should be given distilled water. The density of pure water is one since one milliliter of pure water has a mass of one gram. When selecting liquids to be used in this activity, try to choose some that will be more dense than water and some that will be less. We suggest rubbing alcohol, cooking oil, hand lotion, dish washing liquid, corn syrup, bath oil, juice, cola, diet cola, etc.

VOCABULARY: linear equation, slope, y-intercept, volume, mass, density, rectangular coordinate system, x-axis, y-axis, point

SUGGESTED PATH FOR REMEDIATION: If a computer graphing utility or a graphics calculator is available, students can use these tools to investigate the effects of changing values of m and b in the slope-intercept equation $y = mx + b$ on the resulting graph. Students who are visual learners often develop intuitive knowledge about slope and y-intercept by looking at many examples. Technology is widely available today to make this approach practical and beneficial for a large group of students.

ADDITIONAL RESOURCES: Introductory activities for teaching linear graphing can be found in *Mathematics Curriculum and Teaching Program*. These materials were developed in Australia by Charles Lovitt, Doug Clark, and others. They are available through the National Council of Teachers of Mathematics.

MASS–VOLUME CORRELATION

INTRODUCTION: Equations of lines can be expressed in slope–intercept form

$$y = mx + b,$$

where m is the slope and b is the y–intercept. Many things in every day life are related linearly to one another. One example is the mass and the volume of a liquid. In this activity you will explore this linear relationship.

PURPOSES:

- ✓ Can you find a linear equation that is a mathematical model for the relationship between the mass and volume of a given liquid?
- ✓ What real physical property does the slope of this line represent?
- ✓ What real physical property does the y–intercept of this line represent?

MATERIALS:

scale capable of measuring to the nearest 0.1 gram
graduated cylinder capable of measuring to the nearest 0.1 milliliter
various liquids
graph paper

PROCEDURES:

1. Determine the mass in grams of the empty graduated cylinder and record this in the table on the next page. (The volume of liquid in the empty cylinder is 0 mL.)
2. Pour a small amount of the liquid you are using into the cylinder, perhaps 5 or 10 mL. Record the exact volume in the table on the next page. Use the scale to find the mass of the cylinder and liquid, and record this in the table.
3. Add liquid to the cylinder, measuring volume and mass after doing so. Repeat the process in such a manner that when the cylinder is full you have accumulated about eight to twelve volume and mass measurements. Record all your measurements on the table on the next page.

Meas. No.	Volume	Mass
1	0	
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		

4. Graph the resulting points on a rectangular coordinate system with mass on the y–axis and volume on the x–axis. You may use a computer or a graphics calculator.

OBSERVATIONS:

1. Do all of these points appear to lie on the same straight line?

   Yes, except outlines that indicate measurement error.

2. What appears to be the y-intercept of this line?

   Weight of the empty cylinder

3. What appears to be the slope of this line?

   Depends on the density of the liquid

4. Use the y-intercept and slope to write an equation for the line containing your data points.

   y = density x + b (weight of empty cylinder)

CONCLUSIONS:

1. The slope of a line is defined to be $\frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are any two points on the line. Since the y-axis was used to graph the mass and the x-axis was used to graph the volume, define the slope in terms of mass and volume.

   Mass per unit volume

2. Use a science book to research the definition of density, and write it below.

   Density equals mass per unit volume

3. What does the slope of the line of your graph represent?

   The density of the liquid

4. Use the equation you have written to determine the probable mass of the cylinder when it contains 65 mL of the liquid.

   Answers will vary.

5. Use the equation that you have written to determine what probable volume of liquid you would have to have in your cylinder to have a mass of 58 grams.

   Answers will vary.

6. What physical reality does the y-intercept of your graph represent?

The weight (mass) of the container

7. Compare the results of this experiment for various liquids. Write a paragraph explaining your conclusions.

The denser a liquid, the heavier the weight of an equal unit of volume.

SUGGESTIONS FOR FURTHER STUDY:

- List other things that you suspect to be related linearly. Can you write equations to model these linear relationships? One example is the temperature in degrees Fahrenheit vs. Celsius. Can you think of others?

TEACHER'S GUIDE
THE FINITE SUM OF INFINITELY MANY TERMS

GOAL: To help students understand the sums of infinite series, where they exist.

STUDENT OBJECTIVE:

✓ To use a physical demonstration to suggest the truth of the following formulas:

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\right)=0 \text{ and } \sum_{n=1}^{\infty}\frac{1}{2^n}=1 .$$

GUIDE TO THE INVESTIGATIONS: The concepts prerequisite for this lesson are the definitions of sequence and series and the concept of the infinite. Two difficult concepts are physically modeled in this lesson, the idea of a finite limit of an infinite sequence and the idea that it is sometimes possible to add infinitely many terms of a sequence and get a finite sum. Seeing, however, is believing.

Students will need poster board, construction paper, scissors, and a ruler. This activity may be conducted by groups working individually or collectively. Each group or individual should follow the directions in the Procedures section and record the results in the Observations section. Resulting posters may be displayed in the classroom or in the corridors of the school.

The questions in the Conclusions section may be used to initiate a class discussion following the activity. Students should be encouraged to model other sequences which produce infinite series with finite sums and sequences which will produce series with infinite sums. Examples in each of these categories can be found in Suggestions for Further Study.

VOCABULARY: sequence, series, infinite, finite, limit, sum

SUGGESTED PATH FOR REMEDIATION: Many students have difficulty with the concepts presented in this activity because, while theoretically one can continue cutting each consecutive piece of paper in half, in reality one quickly gets to the point where the size of the paper is so small that it is physically impossible to make the cut. While you can write as many terms of the sequence as you wish or as time and the available paper will allow, you can physically model only about the first five or six terms. This difficulty can be turned to an advantage if the student is encouraged to think about the notation in a more intuitive way. If one tries to keep cutting a piece of paper in half, at some point, which depends on the size paper, the type of cutting instrument, etc., one will have a piece of paper too small to

continue. Thus, $\lim_{n\to\infty}\left(\frac{1}{2^n}\right) = 0$. Likewise, if one tiies to put together the pieces representing a half, a fourth, an eighth, etc., then after a while the total will be so close to one square unit that for all practical purposes the area *is* one square unit. Thus, $\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$. The key to the understanding of these concepts is in making a connection between what one writes and what one means.

ADDITIONAL RESOURCES: *Teaching Mathematics: A Sourcebook of Aids, Activities, and Strategies, Second Edition* (Prentice Hall: 1988) by Max A. Sobel and Evan M. Maletsky includes other examples of models for sequences and series (pp. 166–171).

THE FINITE SUM OF INFINITELY MANY TERMS

INTRODUCTION: Mathematics is a practical, theoretical science often based on actual happenings. This activity lets you look at a real situation and introduces the mathematical notation that is used to represent this real situation.

PURPOSES:

- ✓ What does it mean to say that the limit of a sequence of numbers is zero?
- ✓ What does it mean to say that you can add together infinitely many numbers and get a finite sum?

MATERIALS:

poster board
construction paper
scissors
ruler
glue or tape

PROCEDURES:

1. Cut a 1 by 1 foot square from construction paper. The area of this square is 1 square foot.
2. Fold the square in half and cut along the fold line to produce two rectangles, each with dimensions 1 foot by $\frac{1}{2}$ foot and area $\frac{1}{2}$ square foot. Label one of these rectangles with its area, $\frac{1}{2}\text{ft}^2$.
3. Now take the unlabeled rectangle and fold it in half lengthwise. Cut along the fold line to make two rectangles with dimensions 1 foot by $\frac{1}{4}$ foot and area $\frac{1}{4}$ square foot. Label one of these rectangles with its area, $\frac{1}{4}\text{ft}^2$.
4. Again take the unlabeled rectangle and fold it in half lengthwise. Cut along the fold line to make two rectangles with dimensions 1 foot by $\frac{1}{8}$ foot and area $\frac{1}{8}$ square foot. Label one of these rectangles with its area, $\frac{1}{8}\text{ft}^2$.
5. Repeat this process, reserving one rectangle labeled with its area, and cutting the other in half until it becomes physically impossible to continue. The pieces that you have produced can be used to make a physical representation of the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$. Record the results of each step in the table provided.

   This sequence clearly has a limit of zero because the pieces which model it eventually become so small, their area so close to zero, you cannot continue.

Sequence Term	Dimensions (ft.)	Area (sq. ft.)
1	1 by $\frac{1}{2}$	$\frac{1}{2}$
2	1 by $\frac{1}{4}$	$\frac{1}{4}$
3	*1 by* $\frac{1}{8}$	$\frac{1}{8}$
4	*1 by* $\frac{1}{16}$	$\frac{1}{16}$
5	*1 by* $\frac{1}{32}$	$\frac{1}{32}$
6	*1 by* $\frac{1}{64}$	$\frac{1}{64}$
7	*1 by* $\frac{1}{128}$	$\frac{1}{128}$
8	*1 by* $\frac{1}{256}$	$\frac{1}{256}$
9	*1 by* $\frac{1}{512}$	$\frac{1}{512}$
10	*1 by* $\frac{1}{1\,024}$	$\frac{1}{1\,024}$
11	*1 by* $\frac{1}{2\,048}$	$\frac{1}{2\,048}$
12	*1 by* $\frac{1}{4\,096}$	$\frac{1}{4\,096}$

6. By fitting the pieces back together and taping them onto posterboard you can model the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots$. Since the pieces fit back together to form the original square, whose area is one, you see that the sum of this infinite sequence is 1.

OBSERVATIONS:

1. How many terms of the sequence were you able to model physically before the pieces became too small?

   Answers will vary, about 6 or 7.

2. What factors contributed to the number of terms of the sequence you were able to model?

   The sharpness of the scissors.

   The care of the measurements.

   The size of the first sheet of paper.

CONCLUSIONS:

1. If you had started with a larger square, say one yard by one yard, would you have been able to model more terms of the sequence?

   Yes; but only one or two more.

2. If so what would the sum of the resulting sequence have been in this case?

   The answers will vary, but the average should be closer to one.

SUGGESTIONS FOR FURTHER STUDY:

- Are there other sequences that have a limit of zero? If so, can you write the first few terms?
- Can you model the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$?
- What should the sum of this sequence be?
- If you add the terms of any sequence with a limit of zero, will you always have a finite sum? Can you give examples to support your answer?

TEACHER'S GUIDE
HIT OR MISS — BASEBALL SIMULATION

GOAL: Students will develop an understanding of the reliability of statistical data.

STUDENT OBJECTIVES:

- ✓ To compare results of simulations based on statistical data with results of actual happenings.
- ✓ To evaluate the reliability of statistical data in making predictions.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are a knowledge of probability, ratio, proportion, and percent. This activity can be conducted by each student individually, as a cooperative group project, or by the class as a whole. It is particularly appropriate in the fall of the year during the baseball play-offs and the World Series. Three different methods of simulation are described: the first using dice; the second, a random number table; and the third, numbers randomly generated by a computer or programmable calculator. Each of these simulations can be done independently, so, if no computer or programmable calculator is available, another method of simulation can be used.

VOCABULARY: data, simulation, prediction, probability, random numbers

SUGGESTED PATH FOR REMEDIATION: In order to use the dice to do the simulation, the probabilities that have been computed as decimals to the nearest thousandth need to be converted to the closest fractions with denominators of 36. Students may need help in setting up the appropriate proportions to accomplish this task. Once these probabilities have been written as rational numbers, students must assign each possible outcome of the roll of the dice to represent a single, double, triple, home run, or out. Making a six by six chart showing all the possible outcomes for rolling two dice may be helpful to students.

ADDITIONAL RESOURCES: An excellent source of investigations related to probability is *Quantitative Literacy Series, The Art and Techniques of Simulation* (Dale Seymour Publications).

HIT OR MISS — BASEBALL SIMULATION

INTRODUCTION: Statistics are based on past performance data and are often used to predict future performance. Data from baseball cards can be used as a basis for a simulation of future performance. If you choose a player who is still playing, you can compare the results of the simulation with the results in the next game.

PURPOSES:

- ✓ What are some ways in which data can be used to construct simulations?
- ✓ How do the results of simulations based on data compare with the results of actual happenings?

MATERIALS:

baseball cards
sports section of the newspaper
one or more of the following:
 pair of dice
 random number table
 computer or programmable calculator

PROCEDURES:

1. Choose a baseball player for your project. Use the player's card to find the following information: number of times at bat, number of hits, doubles, triples, and home runs. Calculate the probability of a single (the number of singles is equal to the number of hits less the number of doubles, triples, and number of home runs), a double, a triple, and a home run. For example: a player with 13,037 times at bat, 3,990 hits, 711 doubles, 129 triples, and 158 homers would have a 0.230 probability of a single, a 0.055 probability of a double, 0.010 probability of a triple, and a 0.012 probability of a homer. Record your probabilities in the Observations section.
2. The first simulation will be done by rolling two dice. There are 36 possible outcomes. Write each of the probabilities you found in step one as a fraction with a denominator of 36. Get as close as you can: $\frac{8}{36} = .222...$, which is as close to .230 as you can get with a denominator of 36; $\frac{2}{36} = .0555...$ is as close to .055 as you can get with a denominator of 36; $\frac{1}{36} = .02777...$ is as close to .010 and .012 as you can get with a denominator of 36. Thus, in this example, a roll resulting in a sum of 4 or 9 would represent a single, since the probability of rolling a 4 or 9 is $\frac{8}{36}$. A roll of 3 would represent a double, since the probability of rolling 3 is $\frac{2}{36}$. A roll with a sum of 2 could represent a triple, since the probability of rolling a 2 is $\frac{1}{36}$. A roll with a sum of 12 could represent a home run, since the probability of rolling a sum of 12 is $\frac{1}{36}$. In this example, any

other sum on the dice would represent an out. Record the probabilities for the player you have chosen as fractions with denominators of 36 in the Observations section and record which rolls of the dice will be assigned to which outcome at bat. Roll the dice to simulate the player's next several times at bat. Compare the results of the simulation with your player's performance in the next game that is played.

3. Repeatedly, select three-digit numbers from the random number table to simulate successive at bats. In the example you have been using, 000–229 represents a single; 230–284, a double; 285–294, a triple; and 295–306, a homer. Any one of the numbers 307–999 will represent an out. Set up intervals to use in your simulation based on the data for your particular player. Record these intervals and the results of the simulations in the Observations section. Compare the results of your simulations with actual performance of your player in his next game.

4. If you can program a computer or programmable calculator to generate random numbers, you can use these numbers to simulate your player's performance in the next game in the same manner you used the numbers from the table.

OBSERVATIONS:

1. Record the probabilities, as decimals to the nearest thousandth, of each of the possible at bat outcomes for your player, based on the data from the player's baseball card.

Your Player's Stats

At-bat	Probability
single	*Answers vary.*
double	
triple	
homer	
out	

2. Record each of the probabilities as a fraction with a denominator of 36, and make appropriate correspondence assignments to rolls of two dice.

probability as decimal = $x/36$

x = 36(probability as decimal)

Dice-roll Equivalents for Your Player

At-bat	Probability	Dice-rolls which Approx. Prob.
single		*Answers vary.*
double		
triple		
homer		
out		

Now roll the dice a few times, and record the results in the table below, which also has a column for you to enter the baseball equivalent of your dice rolls. These are the results which you are to compare to your player's next at-bats in actual games.

Simulation Results

Trial	Dice Result	Baseball Equivalent
1		*Answers vary.*
2		
3		
4		
5		

3. Record the assignment of values for three-digit random numbers to the possible at bat outcomes based on the probabilities for your player. Record the outcome of five successive simulations using numbers from a random number table to represent your player's next three at-bats. Use the table below to record your results.

Simulation Results

Trial	Random Number Result	Baseball Equivalent
1		*Answers vary.*
2		
3		
4		
5		

4. Record the assignment of intervals between 0 and 1 representing possible at bat outcomes based on the probabilities for your player. Record the outcome of three successive computer simulations using randomly generated numbers to represent your player's next five at-bats. Use the table below to record your results.

Simulation Results

Trial	Random Number Result	Baseball Equivalent
1		*Answers vary.*
2		
3		
4		
5		

CONCLUSIONS:

1. How do the results of the three different simulation techniques compare to each other?

They should be close to one another.

2. How do the results of the simulations compare to the actual performance of your player?

Answers will vary. We would expect the results to be relatively similar.

3. Discuss the pros and cons of each simulation technique and of simulation in general as a tool for predictions.

Simulations cannot take into account factors like game pressure, health of player, etc.

SUGGESTIONS FOR FURTHER STUDY:

- See if you can get data from team sports at your school to create simulations of performance of various student athletes. Compare the results of these simulations with their actual performance.
- During the World Series choose a player, simulate his performance for each game in the series, and compare it with the results of your simulations.

PROBLEM SOLVING

TEACHER'S GUIDE
A ROOM FULL OF PING PONG BALLS

GOAL: The student will develop an understanding of the concept of ratio and proportion and experience problem solving in a situation where the correctness of the answer cannot directly be validated.

STUDENT OBJECTIVES:

- ✓ To calculate the volume of a ping pong ball.
- ✓ To calculate the volume of the mathematics classroom.
- ✓ To estimate the number of ping pong balls that would fit in the classroom.
- ✓ To find the volume of a shoe box.
- ✓ To find the number of ping pong balls that will fit in the shoe box.
- ✓ To create a proportion to estimate the number of ping pong balls which will fit in the classroom based on the number that fit in the shoe box.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this investigation include estimation skills, an ability to use ratio and proportion, the concept of volume, and the formulas used to compute the volumes of spheres and rectangular prisms. This investigation can be done individually or in cooperative groups. The need for a lot of ping pong balls suggests that groups might be more practical. There is no method of directly validating the results of these calculations. Encourage each group to use a different size shoe box so that comparisons of the various group solutions can be made in an attempt to determine if the size of the shoe box effects the outcome.

VOCABULARY: volume, sphere, rectangular prism, ratio, proportion, estimation, approximation

SUGGESTED PATH FOR REMEDIATION: Some students may need assistance with making measurements and remembering the formula for computing the volume of a sphere. It is probably better to encourage these students to find this formula themselves. Learning to locate needed information is an important part of problem solving.

ADDITIONAL RESOURCES: *Data Analysis and Statistics Across the Curriculum: Addenda Series, Grades 9-12,* by Gail Burrill, John C. Burrill, Pamela Coffield, Gretchen Davis, Jan de Lange, Diann Resnick, and Murray Siegel (National Council of Teachers of Mathematics) gives additional examples of how data collection and analysis can be integrated in the standard high school mathematics curriculum.

A ROOM FULL OF PING PONG BALLS

INTRODUCTION: How many ping pong balls do you think it would take to fill your mathematics classroom? This investigation will attempt to answer that question by using experimentation and ratio and proportion.

PURPOSES:

✓ What is the volume of a ping pong ball?

✓ How many ping pong balls will fit in your mathematics classroom?

MATERIALS:

shoe box
ping pong balls to fill the shoe box
meter stick

PROCEDURES:

1. Use the meter stick or ruler to measure the diameter of a ping pong ball. Check the container that the ball came in to see if it gives the dimensions of the ball. Use this information to calculate the volume of the ping pong ball.
2. Use the meter stick to measure and calculate the volume of your mathematics classroom. If your classroom is not perfectly rectangular you may want to divide the room into rectangular regions, calculating the volume of each region, and adding them to find the volume of the entire room.
3. Find the dimensions of the shoe box, and calculate its volume.
4. Fill the shoe box with ping pong balls, and record the number used. Be sure that the lid will fit evenly on the box.

OBSERVATIONS:

1. What is the diameter of the ping pong ball to the nearest tenth of a centimeter?

   3.8 cm

2. What is the volume of a ping pong ball calculated in cubic centimeters?

   ≈ 28.73 cm³

3. What is the volume of your classroom in cubic centimeters?

   Answers will vary.

4. Divide the volume of the room by the volume of a ping pong ball, to estimate the number of ping pong balls that will fit in the room.

   Answers depend on room size.

5. What is the volume of the shoe box in cubic centimeters?

   Answers depend on the box used.

6. Divide the volume of the shoe box by the volume of the ping pong ball to estimate the number of ping pong balls that will fit in the box.

   Answers depend on the box used.

7. How many ping pong balls were you able to get in the shoe box?

   It should be less than the estimate in #6.

CONCLUSIONS:

1. Do you believe that the estimate of the number of ping pong balls which will fit in your classroom (arrived at by dividing the volume of the room by the volume of a ping pong ball) is too large or too small? Explain?

   Too large—it does not take into account the space lost due to the shape of the balls.

2. Write a proportion with the ratio of ping pong balls to the volume of the shoe box equal to the ratio of the number of ping pong balls that will fit in the classroom (the unknown) to the volume of the room. Solve this proportion to get an estimate for the number of ping pong balls that will fit in your mathematics classroom.

   $$\frac{\textit{\# of balls in box}}{\textit{volume of box}} = \frac{x}{\textit{volume of room}}$$

   $$x = \frac{\textit{volume of room (\# of balls in box)}}{\textit{volume of box}}$$

3. Compare the two estimates. Which do you think is more accurate? Why?

The second estimate is smaller than the first and probably more accurate, because it does take into account the space lost due to the shape of the balls.

SUGGESTIONS FOR FURTHER STUDY:

- Get a larger box, such as the boxes used to deliver the copying paper for the school. Find the volume of this box. Write a proportion with the shoe box to estimate the number of ping pong balls required to fill the larger box. Actually fill the larger box with ping pong balls, and count them. How does the actual solution compare with the solution arrived at by using the proportion? What is the percent of error? Can you use this information to adjust your estimate for the number of ping pong balls that the room will hold?
- Find out the amount of the national debt. If this amount of money in hundred dollar bills was stacked, how tall would the stack be?

TEACHER'S GUIDE
DISTANCE vs. TIME

GOAL: The student will develop an understanding of the concept of modeling physical events with quadratic equations.

STUDENT OBJECTIVES:

- ✓ To study the relationship between distance and time when acceleration is occurring.
- ✓ To write a quadratic equation to serve as a mathematical model for this relationship.

GUIDE TO THE INVESTIGATIONS: As a prerequisite for this investigation, students need to know about quadratic functions, especially those of the form $y = ax^2$. They will need to work cooperatively to collect data for this investigation. A starter and eleven timers are required. If eleven stop watches are not available, then the data will have to be collected in a different manner. (Try asking the coaching staff to loan you stopwatches.)

The twelve foot long, two by four board used in this investigation will have to be prepared to allow for the tennis ball to travel its length without going over the edge. There are several ways this can be accomplished. One method is to ask a wood craftsman to rout out a trough for the ball to roll down. Another method not needing special tools or skills is to tack a narrow, raised wooden strip along both sides of the two by four. The final step in preparing the device is to paint marks at one foot intervals along the path so that the timers can easily see when the ball reaches the distance for which they are timing.

VOCABULARY: distance, time, acceleration, mean, rectangular coordinate system, x-axis, y-axis, quadratic function, linear function

SUGGESTED PATH FOR REMEDIATION: Students sometimes have trouble collecting accurate data which enable them to predict the equation accurately. The goal is to find an equation that provides a reasonable (not exact) mathematical model for this situation. Textbooks tend to give examples where the data and the mathematical model are totally congruent. Mathematical models, however, are always approximations to reality. Trial and error or guess-test-revise strategies are valid methods for modeling these data. Graph the equation $y = ax^2$ with $a = 1$, and compare the data points to the graph. Based on what you see, change the value of a, and compare again. Repeat this process until a value for the coefficient is found that provides a reasonably good match with the data points. If students have trouble making adjustments to the coefficient, you might give a brief review of the effects of positive and negative coefficients and of increasing and decreasing $|a|$.

ADDITIONAL RESOURCES: In *Precalculus Mathematics: A Graphing Approach* (Addison-Wesley), there is a section entitled "Quadratic Functions and Geometric Transformation," which is an excellent resource for techniques to help students develop intuition about graphs of quadratic equations. This book was developed by the Ohio State University Calculator and Computer Precalculus Project, directed by Bert Waites and Frank Demanea.

DISTANCE vs. TIME

INTRODUCTION: Acceleration is the change in velocity per unit of time. Velocity is the change in distance per unit of time. While feet per second expresses velocity, feet per second per second is a measure of acceleration.

PURPOSE:

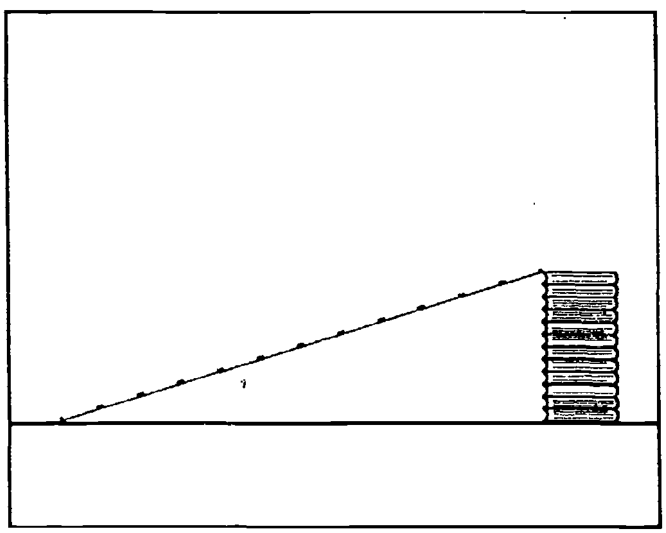
✓ What is the relationship between distance and time when acceleration is occurring?

MATERIALS:

one prepared twelve foot long, two by four board (the runway)
tennis ball
eleven stop watches
graph paper

PROCEDURES:

1. Elevate one end of the runway to provide an inclined path down which to roll the tennis ball. You may prop it securely on a stack of books, but be careful not to move it or change the elevation for the duration of the experiment.



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2. Twelve people are required to collect data. One person will stand at the top of the inclined path and release the ball. When the ball is released, this person will say, "Go," and the eleven people stationed down the inclined path at one-foot intervals will begin timing. When the ball reaches the mark by which each is stationed, he or she will stop timing and record the time elapsed. It may be wise to do a trial run or two to allow each experimenter to become somewhat comfortable with the process. Because of possible inaccuracy in the time measures, repeat the experiment three times and find the average of the three readings for each of the eleven time measures. Record your data in the table on the next page.

Distance feet	Time (seconds) Trial One	Time (seconds) Trial Two	Time (seconds) Trial Three	Time (seconds) Mean
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				

3. Make a graph with time plotted on the x-axis and distance plotted on the y-axis. Note that the point (0, 0) is on the graph because, when the ball was at the top of the incline, no time had lapsed since its release. The points on this graph will appear to lie on one branch of a parabola with equation $y = ax^2$. Use trial and error to find the value of a giving the parabola which most closely fits this set of data points.

4. Make another graph with time squared plotted on the x-axis and distance plotted on the y-axis. The points on this graph will appear to lie in a line with the equation $y = mx$. Find the coefficient of x (slope) giving the line that most closely fits this set of data points.

OBSERVATIONS:

1. What quadratic function did you decide on as a mathematical model for these data?

   Answers will vary, depending on the angle of inclination, etc. $y = ax^2$

2. What linear equation did you decide on as a mathematical model for the graph of distance versus time squared?

   Answers will vary. $y = ax$

CONCLUSIONS:

1. Explain why the graph of time versus distance is not linear.

   The ball's acceleration increases with time.

2. Explain the relationship between the coefficient of x^2 in the time versus distance graph and the coefficient of x in the time squared versus distance graph.

They should be the same.

3. If the elevation of the two by four is increased or decreased, how will it effect the coefficient of x^2 in the equation of the resulting graph?

The greater the elevation, the larger the magnitude of the coefficient.

4. What effect does gravity have on this experiment?

Gravity makes it possible by providing the downward force which causes acceleration.

5. Compare the actual time-distance data with the times that result by substituting the distances into the equations you derived.

Distance feet	Time Actual	Time Quadratic Equation	Time Linear Equation
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			

6. Use the equation of the time versus distance graph to predict the number of seconds required for the ball to travel four and one-third feet.

Answers will vary.

SUGGESTIONS FOR FURTHER STUDY:

- Repeat this experiment using various heights for the incline. Is there a relationship between a and the height of the incline? What is it? Given the height of the incline, can you predict the value of a?
- What would happen if you repeated this experiment on the moon?

TEACHER'S GUIDE
VCR COUNTERS

GOAL: The student will develop an understanding of how quadratic equations can be used to model real world situations.

STUDENT OBJECTIVES:

- ✓ To collect data showing the lapsed play time for a VCR and the corresponding counter reading.
- ✓ To find a linear or a quadratic function that is a reasonably good model for the relationship between the counter reading and the lapsed play time.
- ✓ To use this equation to predict lapsed play times from counter readings.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this investigation are an ability to plot points on the Cartesian coordinate system, graph linear or quadratic functions of the forms $y = ax$ or $y = ax^2$, and work with the formula for a circle.

Video Cassette Recorders are fairly common in most homes, so it may be possible for students to collect this data at home as an out of class assignment. This may prove popular if the tape they are playing is a good movie.

Students can work alone or in cooperative groups. It is desirable to have each group collect data using a different VCR so the groups can compare results and decide if one formula is accurate for all VCRs.

Some VCR's provide quadratic data, while newer ones seem to provide linear data. It will be interesting to see if students provide samples of each type.

VOCABULARY: quadratic function, x-axis, y-axis, data, radius

SUGGESTED PATH FOR REMEDIATION: If students do not understand graphing linear and quadratic equations of the forms $y = ax$ or $y = ax^2$, they may have trouble when they are trying to find an equation to fit their data. One approach to graphing equations of quadratic form is to plot the vertex of the parabola, $(0, 0)$. From the origin, move right *1* unit and up a units, then from the origin move right *2* units and up $4a$ units, then from the origin move right *3* units and up $9a$ units, and so on. This yields the right branch of the parabola, the portion for which this problem is defined. If desired, the left branch can be drawn using symmetry.

ADDITIONAL RESOURCES: *Mathematical Modeling in the Secondary School Curriculum*, edited by Frank Swetz and J. S. Hartzler (National Council of Teachers of Mathematics) is an excellent resource for additional mathematical modeling problems.

VCR COUNTERS

INTRODUCTION: Most VCRs have a counter which increases as the tape is played, but it really does not tell you much. If the counter goes from 0 000 to 1 000, 0 500 is not necessarily halfway through the tape. The purpose of this investigation is to collect data from which to develop a formula for relating your VCR's counter reading to the amount of time the tape has been running.

PURPOSE:

✓ Can a formula be developed to relate the time a VCR tape has been running and the difference between the beginning and current counter readings?

MATERIALS:

VCR
a T120 VCR tape
digital clock or stopwatch
graph paper

PROCEDURES:

1. On a T120 tape, record some programs you will enjoy watching. Fill the tape to the very end. If your VCR has variable speeds, use the fastest. Place the tape in the VCR, rewind the tape to the beginning, and reset the counter to 0000. **Do not use a commercially recorded movie.**
2. Press play on the VCR and note the time on the clock, or start the stopwatch.
3. Play the tape and record the counter reading and the lapsed time every few minutes. Run the tape all the way to its physical end as you record data.
4. Plot these data on a graph with the counter reading on the x-axis and the lapsed time on the y-axis.

OBSERVATIONS:

1.

Record of Observed Data

Elapsed Time Minutes	Counter Reading

2. Look at a graph of these data. An examination of this scatter plot may reveal that the relationship between the counter reading and lapsed time is not linear. Is it quadratic? Can you find through trial and error a quadratic equation of the form $y = ax^2$ which fits the data reasonably well? If it is linear, can you find an equation of the form $y = ax$ which best fits the data?

Answers will vary.

CONCLUSIONS:

1. Assuming that you began the counter at 0000, use the equation that you wrote to predict the amount of time a tape has played in your VCR when the counter reads, 1 340, 1 500, and 0 350. Play your tape and see if your equation was at all accurate.

Answers will vary.

2. Do you think that your formula works for all VCRs? If you record two movies on a tape and label it with the beginning counter number for the second movie, when you take the tape to a friend's house to watch the second movie, will you be able to find its beginning exactly from the label?

The formula may (probably) vary between VCR's.

SUGGESTIONS FOR FURTHER STUDY:

- Repeat the experiment recording at a different speed, and compare your results. Repeat it at the same speed with a T160 tape, and compare your results.
- If there is an electronic repair expert in your community, it would be interesting to have this person come in as a resource and explain the operations of the counters on VCRs.
- The Texas Instruments TI – 81 programmable, graphics calculator has the ability to produce a regression equation using power regression. The result of this statistical procedure can be compared to the results of your investigation. (If power regression is to be used, do not enter the data point (*0*, *0*) because it will result in the TI – 81's trying to divide by zero.).

TEACHER'S GUIDE
HOW MUCH DOES THIS BOX HOLD?

GOAL: The student will develop an understanding of the concept of maximization and use equations to model a real world problem.

STUDENT OBJECTIVES:

✓ To construct an open-top box from a sheet of paper.

✓ To find the maximum volume possible for a box constructed, by removing a square from each corner of a 19 by 26 inch sheet of paper.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are knowledge of the concept of volume, volume formulas, and the willingness to try setting up and solving real problems. This activity works very well as a cooperative group activity. Each group member can construct a different box until all nine suggested boxes have been built. The boxes produced in this activity can be displayed in the classroom or school library.

VOCABULARY: rectangular prism, volume, height, length, width, maximum, square

SUGGESTED PATH FOR REMEDIATION: This is an excellent opportunity to help students internalize the formula for the volume of a rectangular prism. Use base ten blocks to cover the bottom of one of their boxes, and ask them, "How many square centimeters did it take to cover the bottom of the box"? Suggest that students multiply the width times the length of the box and compare this with the number of square centimeters required. Finish by filling the box with cubic centimeters. How many layers of blocks were required to fill the box? Compare this with the height of the box. How many cubic centimeters did it take to fill the box? Compare this with the product of the width, length, and height. This should help the students understand the formula for volume of the rectangular prism.

ADDITIONAL RESOURCES: Other investigations using mathematical modeling can be found in *Mathematical Modeling in the Secondary School Curriculum*, by Frank Swetz and J. S. Hartzler (National Council of Teachers of Mathematics).

HOW MUCH DOES THIS BOX HOLD?

INTRODUCTION: A certain manufacturing process produces a liquid by-product. The company wishes to build some vats to store this by-product. They are to be in the form of rectangular boxes with open tops, made from sheets of metal 19 by 26 feet, one vat from each sheet. A metal works will be contracted to build the vats by cutting a square out of each of the corners of the metal sheet, folding up the sides, and welding the corner edges. If you own the metal works, you must convince the manufacturer that you can build the vat that has the greatest volume from the fixed-size metal sheets. This investigation will help you solve this problem.

PURPOSES:

✓ Can you construct a rectangular container from a sheet of material by cutting a square out of each corner and folding up the sides?

✓ If such a container is built from a 19 by 26 foot flat sheet, what size square should be cut out of the corners to produce the container with the greatest volume?

MATERIALS:

centimeter grid paper measuring 19 by 26 cm
scissors
tape
ruler
calculator

PROCEDURES:

1. From each corner of a 19 by 26 cm sheet of grid paper, cut a 1 cm square. Fold up the sides and tape the corner edges together to form a rectangular open-top box. Record the dimensions and volume of this box in the table provided in the Observation section.

2. Repeat this process eight more tim‹ 3, cutting squares of 2, 3, 4, 5, 6, 7, 8, and 9 cm, respectively.

OBSERVATIONS:

1. Record the information about your boxes in the table on the next page.

Size of Square cm	Box Length cm	Box Width cm	Box Height cm	Box Volume cubic cm
1 x 1	24	17	1	408
2 x 2	22	15	2	660
3 x 3	20	13	3	780
4 x 4	18	11	4	792
5 x 5	16	9	5	720
6 x 6	14	7	6	588
7 x 7	12	5	7	420
8 x 8	10	3	8	240
9 x 9	8	1	9	72

2. Make a graph with the lengths of the sides of the squares cut out on the x-axis and the volumes of the resulting boxes on the y-axis.

3. Which one of these boxes has the greatest volume?

   18cm • 11cm • 4cm

CONCLUSIONS:

1. Why does the table stop with a square of 9 cm on each side?

   It would not be possible to cut 10cm squares.

2. How is the measure of the side of the square related to the height?

   It equals it.

   The width?

   Width = 19cm – 2x the measure.

   The length?

   Length = 26cm – 2x the measure.

   The volume?

   Volume = the product of the three above.

3. Do you think that a box with a greater volume could be produced if the length of the side of the square were not an integer? If so, what values do you have in mind?

   Perhaps trial and error with a calculator will reveal an answer.

   For example: 3.5cm x 3.5cm square produces a volume of $798cm^3$

4. Use the formula for the volume found in Question 1 and the numbers that you identified in Question 2, to see if you can find a way to build a box with greater volume. You need only find the answer to the nearest tenth of a centimeter. Use a sheet of 19 by 26-cm grid paper, your ruler, tape, and scissors to build the box you find.

   3.6cm x 3.6cm produces a volume of 798.624cm^3

SUGGESTIONS FOR FURTHER STUDY:

- If you know differential calculus, use the first derivative test to find the maximum of the volume function.
- This activity can be extended to a discussion of minimizing surface area and maximizing volume, a process sometimes used in designing packaging for commercial products.

TEACHER'S GUIDE
PARAMETRIC EQUATIONS

GOAL: To develop students' ability to use parametric equations to model paths of projectiles.

STUDENT OBJECTIVE:

✓ To write parametric equations modeling the path of a projectile.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity include knowledge of graphing points in the plane, the cosine and sine functions, and the effect of gravity on airborne objects.

The experiment described in this activity can be done by the whole class or by cooperative groups. The instructions for building the necessary mechanism for this experiment are included in the Procedures section. The materials required include a six by eighteen inch piece of half-inch plywood, a wooden foot ruler, glue, two nails, a heavy rubber band, a metal nut, a yard stick, a protractor, a stop watch, and a graphics calculator, such as the Texas Instruments TI-81, or a computer with graphing software. This activity can be done without access to the hi-tech devices; however, plotting these graphs by hand is very time consuming and not very accurate.

VOCABULARY: cosine, sine, angle of inclination, distance, initial velocity, gradational force, abscissa, ordinate

SUGGESTED PATH FOR REMEDIATION: A review of the sine and cosine functions and polar coordinates with the unit circle may be necessary for some students. For example: ($\cos 20°$, $\sin 20°$) represents a point on the unit circle with center (0, 0) such that the angle formed by the positive horizontal axis and the line from (0, 0) to ($\cos 20°$, $\sin 20°$) measures 20°. The point ($a \cos 20$, $a \sin 20$) is on a circle with radius a and center (0, 0) such that the angle formed by the positive horizontal axis and the line from (0, 0) to ($a \cos 20$, $a \sin 20$) measures 20°.

Some students may also need additional explanation of the effect of gravity. Acceleration due to gravity is – 32 feet per second; velocity due to gravity is a function of the time in seconds the object has been falling ($-32t$); and gravity's effect on the height of a falling object is given by the function ($-16t^2$) where t is the time in seconds the object has been in flight. (Though it is not necessary for this exercise, you may remember that the velocity function is the derivative of the position function, and the acceleration function is the derivative of the velocity function.)

ADDITIONAL RESOURCES: *College Algebra and Trigonometry: A Graphing Approach, Second Edition*, Demana, Waits, and Clemens, (Addison-Wesley, 1992) has several good application problems involving parametric functions. See Chapter 11. An example involving baseball, which may interest student athletes, is on page 540.

PARAMETRIC EQUATIONS

INTRODUCTION: Parametric equations may be used to specify an (x, y) location in a plane at a given time, t. Thus $x(t)$ gives the location's abscissa, x-coordinate, as a function of t; while $y(t)$ gives the location's ordinate, y-coordinate, as a function of t.

PURPOSE:

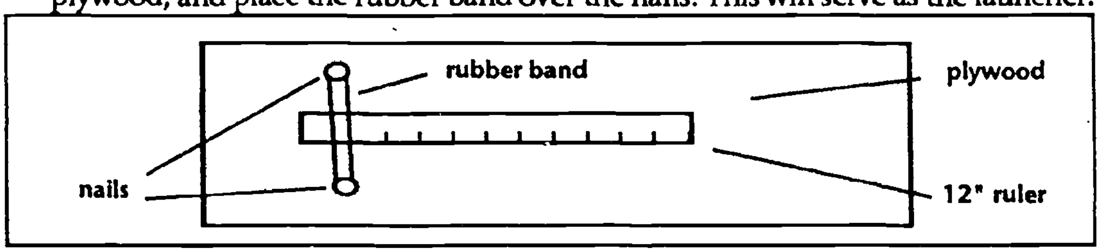
✓ Can parametric equations be used to determine the path of a projectile?

MATERIALS:

a 6 by 18 inch piece of half-inch thick plywood
a wooden foot ruler
glue
two nails
a heavy rubber band
a metal nut
a yard stick
a protractor
a stop watch
a graphics calculator or a computer with graphing software

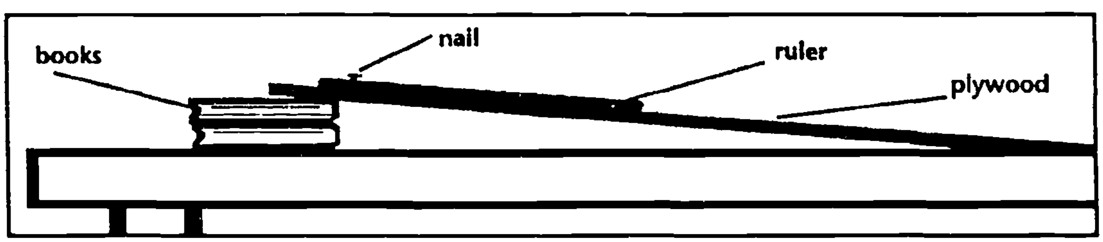
PROCEDURES:

1. Consult illustration 32-1 to nail the two nails in the plywood, glue the ruler to the plywood, and place the rubber band over the nails. This will serve as the launcher.



32-1

2. Prop the launcher on a table at an angle as shown in illustration 32-2. Use a brick or a stack of books to elevate the end with the nails and rubber band.



32-2

3. Use the protractor to measure the angle of inclination and the yard stick to measure the distance from the floor to the launcher. Record this information in the chart in the Observation section.
4. Pull the rubber band back. Use the ruler glued to the launcher to measure how far back you pull it.
5. Taking care that no object and no person is in the line of fire, place the nut in the launcher and fire. Repeat this process several times, always pulling the rubber band back the same distance. In each trial, use the stop watch to measure the time the nut is in the air. Measure the distance the nut travels each time using the yard stick. Record this information in the table in the Observations section. The times and distances should be fairly uniform for each trial.
6. Average the travel times and distances traveled for all the trials. Enter these averages in the table in the Observations section.

OBSERVATIONS:

1. Complete the table below.

The angle of inclination of the launcher to the horizontal is_______degrees and the height of the launcher from the floor is___________inches. The rubber band was pulled back___________inches.

No. of trials	Time (sec.)	Distance (ft.)
1st		
2nd		
3rd		
4th		
5th		
6th		
7th		
8th		
9th		
Average		

CONCLUSIONS:

1. The path of the nut can be modeled mathematically using the parametric equations:
$x(t)$ = the initial velocity of the nut times t times the cosine of the angle of inclination; and
$y(t)$ = the initial velocity of the nut times t times the cosine of the angle of inclination minus $16t^2$ plus the height of the launcher above ground.

Use great caution to keep all values and measurements in these formulas in the same units. As written, the formulas expect all lengths to be in feet and all times in seconds.

The $-16t^2$ represents the effect of gravity on the height of the nut, and it is given in feet. You know the height of the launcher above the ground and the angle of inclination. Use the symbol v_0 to represent the initial velocity and write the set of parametric equations that fit the data you gathered from your experiment.

$x(t)$ = *Answers vary.*

$y(t)$ = *Answers vary.*

2. Let's call the average time it took for the nut to hit the ground a. You know that $x(a)$ is the average distance in feet traveled by the nut and $y(a) = 0$ because at time $t = a$, the nut is on the ground! Use this information to calculate the initial velocity of the nut.

Answers depend on data, angle, and height.

3. Use the data you collected to write the complete parametric equations giving the location of the nut during it's flight.

$x(t)$ = *Answers vary.*

$y(t)$ = *Answers vary.*

4. Graph this set of parametric equations using a graphics calculator or computer with graphing software. What is the maximum height achieved by the nut during its flight?

5. What relationship does this experiment have to firing missiles?

The distance a missile will travel, and the height to which it will rise, depend in part on its velocity, the angle at which it is fired, and the height from which it is fired.

SUGGESTIONS FOR FURTHER STUDY:

- Repeat the experiment varying the angle of inclination, the distance that you pull the rubber band back, and the height of the launcher. How do these changes effect the path of the nut?
- How are vectors related to this experiment?

TEACHER'S GUIDE
BUILDING THE CAN

GOAL: The student will develop an understanding of the concept of the maximum value of a function.

STUDENT OBJECTIVES:

- ✓ To build cylinders with varying radii but fixed distance (20 cm) from the top to bottom of the material sheet from which the cylinder pattern may be cut.
- ✓ To use these models to develop a formula for the volume of a cylinder cut from material with a fixed height.
- ✓ To use this formula and the information discovered by examining the models to find the radius to the nearest tenth of a centimeter which will produce the cylinder with the greatest volume, given that the distance from the top to the bottom of the pattern must be 20 cm.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this investigation are the formulas for area of a circle, volume of a cylinder, and the ability to write algebraic expressions for real problems.

This investigation can be done individually or by students working in cooperative groups. The models can be displayed on a table in the classroom or mounted on a bulletin board. The activity can be varied by choosing another value for the fixed distance from the top to the bottom of the pattern.

VOCABULARY: cylinder, volume, radius, circle, π, circumference, maximum

SUGGESTED PATH FOR REMEDIATION: This activity provides an excellent opportunity to review with students who do not understand the concept of volume or the formula for the volume of the cylinder. Remove the top from one of the cylinders, and use modeling clay to construct a 1 cm thick disc that will cover the bottom. Discuss the volume of this clay disc by comparing it to a flat from a set of base ten blocks. Discuss how many cubic centimeters would be used to cut from the flat a disc the size of the one made from clay. Compare this estimate with the result of the πr^2 calculation. Discuss the number of these discs required to fill the cylinder, and use this number to estimate the number of cubic centimeters it would take to fill the cylinder. Compare this with the result of the $\pi r^2 h$ calculation. This activity provides a good chance to assist students who have been having difficulty in formulating algebraic expressions for story problems.

ADDITIONAL RESOURCES: *Exploratory Problems in Mathematics* by Frederick Stevenson (National Council of Teachers of Mathematics) contains many open-ended problems providing additional opportunities to conduct mathematical explorations.

BUILDING THE CAN

INTRODUCTION: The pattern in illustration 33-1 on the next page can be cut out and assembled to form a cylinder. The circumference of the circles must be equal to the length of the rectangle. Thus the length of the rectangle must be equal to $2\pi r$, where r is the radius of the circular top and bottom. The dimension of the rectangle which becomes the height can vary independently of the radius of the circle. In this investigation you will examine the volume of the can, if the distance from the bottom of the pattern to the top of the pattern is fixed at 20 cm, while the radius of the circle may vary.

PURPOSES:

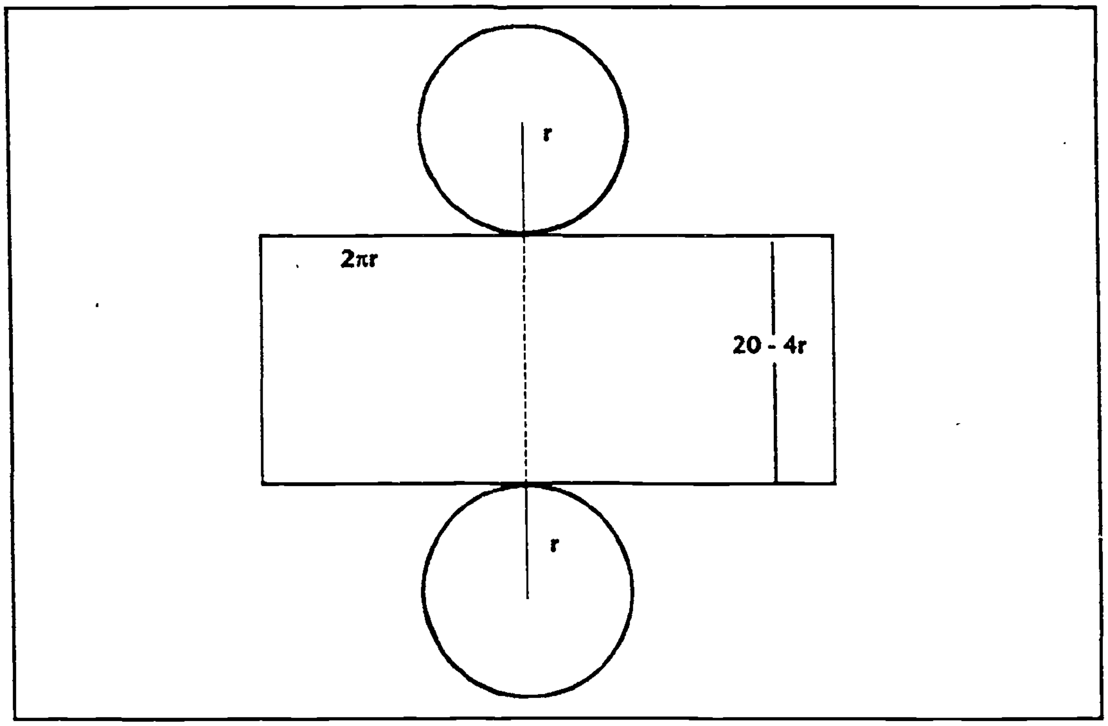
- ✓ If the distance from the top to the bottom of this pattern is fixed at 20 cm, but the radius is allowed to vary, how does changing the length of the radius change the volume of the resulting cylinder?
- ✓ What radius should be chosen if you are to maximize the volume of the can?

MATERIALS:

compass
ruler
construction paper
scissors
tape
calculator

PROCEDURES:

1. Cut the construction paper to 20 x 26 cm.
2. Construct the pattern for the cylinder so that the radius of the circles at the top and bottom is 1 cm and the total distance from the top *of the pattern* to the bottom is 20 cm. Label the body of the cylinder "radius 1 cm". Cut out this pattern and tape it to form a cylinder.
3. Construct the pattern for the cylinder so that the radius of the circles at the top and bottom is 2 cm and the total distance from the top *of the pattern* to the bottom is 20 cm. Label the body of the cylinder "radius 2 cm". Cut out this pattern and tape it to form a cylinder.
4. Construct the pattern for the cylinder so that the radius of the circles at the top and bottom is 3 cm and the total distance from the top *of the pattern* to the bottom is 20 cm. Label the body of the cylinder "radius 3 cm". Cut out this pattern and tape it to form a cylinder.



33 - 1

5. Construct the pattern for the cylinder so that the radius of the circles at the top and bottom is 4 cm and the total distance from the top *of the pattern* to the bottom is 20 cm. Label the body of the cylinder "radius 4 cm". Cut out this pattern and tape it to form a cylinder.

OBSERVATIONS:

1. Fill in the table below.

Radius of Base	Area of Base	Height of Pattern	Volume
1 cm	*≈ 3.14cm²*	*16cm*	*≈ 50.27cm³*
2 cm	*≈ 12.57cm²*	*12cm*	*≈ 150.80cm³*
3 cm	*≈ 28.27cm²*	*8cm*	*≈ 226.19cm³*
4 cm	*≈ 50.27cm²*	*4cm*	*≈ 201.06cm³*

2. Which of these cylinders has the greatest volume?

3cm radius.

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CONCLUSIONS:

1. Can you build a cylinder with radius 5 cm with the distance from the top to the bottom of the pattern fixed at 20 cm?

   No. The top and bottom would be too big, having a height of 0.

2. If you used values for the radius that were not whole numbers, do you think you could get more volume? If you have access to a calculator, try a few.

   Trial and error with a calculator will yield an answer.

   No. 3.3cm radius produces 232.64cm^3.

3. Let *r* be the radius. Write formulas which express each of the following in terms of *r*:
   the height of the cylinder;

   $20 - 4r$

   the area of the circular base;

   πr^2

   the volume of the cylinder.

   $(20 - 4r)(\pi r^2)$

4. Use the formula to compute the volume for values of the radius which may result in a cylinder of greater volume than you have already found. See if you can find the radius to the nearest tenth of a centimeter that will produce the greatest volume. Construct a pattern, and build this cylinder.

   3.3cm radius produces the maximum volume if measures to the nearest .1cm are allowed.

5. Write an explanation of why the cylinder volume increases with the increase in radius up to a point and decreases from that point. Note that the volume of the cylinder is zero when $r = 0$ and when $r = 5$.

   The volume depends on the height, which decreases as r increases, and the area of the base, which increases as r increases.

SUGGESTIONS FOR FURTHER STUDY:

- If you know differential calculus, use the first derivative test to find the maximum of the volume function. Compare this with the result of the investigation.
- Investigate how the surface area of the cylinder changed as the radius varied.

TEACHER'S GUIDE
MODELING THE DIFFERENCE OF SQUARES

GOAL: The student will develop an understanding of factoring patterns for the difference of two squares and trinomials which are perfect squares.

STUDENT OBJECTIVES:

- ✓ To produce a geometric model showing the factoring pattern for the difference of squares.
- ✓ To produce geometric models showing the factoring patterns for perfect-square trinomials.

GUIDE TO THE INVESTIGATIONS: The prerequisite for this activity is an ability to use variables to represent numerical values and skill in multiplying and factoring polynomials. This investigation is a hands-on activity to help students conceptualize two of the most common factoring patterns. The models can be displayed in the classroom or in other appropriate locations in the school.

VOCABULARY: difference of squares, perfect-square trinomial, factor, area

SUGGESTED PATH FOR REMEDIATION: Students having trouble with factoring patterns for the difference of squares and perfect-square trinomials can use computer spread sheets to explore the relationships between the two sides of the formulas. For example, to explore the difference of squares pattern, label the columns a, b, a^2, b^2, $a^2 - b^2$, $a - b$, $a + b$ and, $(a + b)(a - b)$. Students may choose numbers for a and b and examine the results in the other columns. When they realize that the $a^2 - b^2$ column and the $(a - b)(a + b)$ column always have the same value, they are making progress toward conceptual understanding.

ADDITIONAL RESOURCES:

Activities for Implementing Curricular Themes from the Agenda for Action, edited by Christian Hirsch (National Council of Teachers of Mathematics: 1986) contains a section entitled "Finding Factors Physically."

MODELING THE DIFFERENCE OF SQUARES

INTRODUCTION: This investigation is designed to demonstrate geometrically the factoring pattern for the difference of two squares.

PURPOSE:

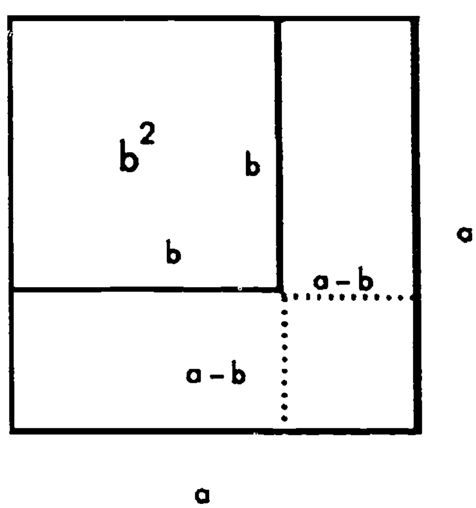
✓ Can a geometric model be found for the factoring pattern for the difference of two squares?

MATERIALS:

construction paper – 2 different colors
ruler
protractor
scissors
marker

PROCEDURES:

1. Use the ruler and protractor to draw a 6-inch square on construction paper. This will represent a^2.
2. Using a different color construction paper, draw a 4-inch square. This will represent b^2.
3. Tape the b^2 onto the a^2 as shown in illustration 34-1 below, the region not covered is $a^2 - b^2$, or the difference of the two squares. Use a marker to separate the $a^2 - b^2$ area into three sections as shown.



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OBSERVATIONS:

1. What is the area of each of the three sections:

   top left area?

   b^2

   top right area?

   $(a - b)(b)$

   bottom left area?

   $(a - b)(b)$

   bottom right area?

   $(a - b)^2$

2. Write $a^2 - b^2$ as a sum of these three areas.

   $2(a - b)(b) + (a - b)^2$

   $(a - b)2b + (a - b)(c - b)$

   $(a - b)(2b + a - b)$

   $(a - b)(a + b)$

CONCLUSIONS:

1. What are the factors of $a^2 - b^2$?

   $a + b$ and $a - b$

2. Use a similar procedure to construct a model showing $(a + b)^2 = a^2 + 2ab + b^2$.
3. Use a similar procedure to construct a model showing $(a - b)^2 = a^2 - 2ab + b^2$.

SUGGESTIONS FOR FURTHER STUDY:

- Expand this model into three dimensions, producing models for the sum and difference of cubes.
- Use algebra tiles to examine factoring patterns for quadratic functions in general.

TEACHER'S GUIDE
CONSTRUCTING TRIG TABLES

GOAL: The student will develop an understanding of the definitions of trigonometric functions, the relationships among these functions, and the ratios of the sides of right triangles.

STUDENT OBJECTIVES:

- ✓ To use drawings of right triangles to find the values of the six trigonometric functions for angles which measure 15, 30, 45, 60, and 75 degrees.
- ✓ To compare the measures computed from these drawings to the values given by a calculator or a table of trigonometric values.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity include the ability to measure angles with a protractor and line segments with a ruler. Students also need a rudimentary understanding of converting fractional ratios to decimal approximations. Students may work independently or in cooperative groups. Be sure that they are reasonably accurate when making angle and length measurements; measurement error may be compounded when the ratios are calculated.

VOCABULARY: angle, line, ray, line segment, degree, ratio, right angle, right triangle, sine, cosine, tangent, cosine, cosecant, cotangent

SUGGESTED PATH FOR REMEDIATION: Students may need additional help when making measurements. They may read the wrong scale on the protractor, incorrectly align the center of the protractor with the vertex of the angle, or inaccurately measure fractional parts of an inch.

ADDITIONAL RESOURCES: Logo computer software may be used to construct the figures which this activity requires.

CONSTRUCTING TRIG TABLES

INTRODUCTION: The purpose of this investigation is to use right triangles to develop a table of trigonometric values for some common angles.

PURPOSES:

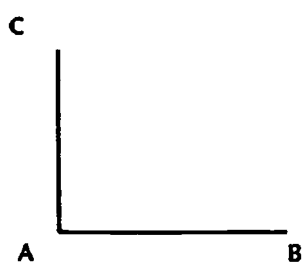
- ✓ Can a table of trigonometric values be developed by constructing a series of right angles?
- ✓ How do the numbers in a table compiled in this way compare with values found in trigonometric tables printed in mathematics texts?
- ✓ How do the numbers in a table constructed in this way compare with values given by calculators?

Materials

ruler
protractor
butcher paper
calculator
table of trigonometric values

PROCEDURES:

1. Use a protractor to draw a right angle on a piece of butcher paper. Label the angle *CAB*. (You will need to make the line segments fairly long. You may want to read through the Procedures before beginning, to see why.)



2. Use the ruler to mark off line segment AX_1 on ray AB, so that the measure of line segment AX_1 is 1 inch. Use the protractor to draw an angle that measures 15° with segment X_1A as one of its sides. Label the point where the other side of this angle crosses line AC as Y_1.

3. Use the ruler to mark off a line segment AX_2 on ray AB, so that the measure of line segment AX_2 is 2 inches. Use the protractor to draw an angle that measures 15° with segment X_2A as one of its sides. Label the point where the other side of this angle crosses line AC as Y_2.

4. Continue this procedure until you have located points X_n and Y_n for $n = 1, 2, 3, 4, 5, 6$.
5. Repeat this procedure for a 30°, a 45°, a 60°, and a 75° angle.

OBSERVATIONS: ***Numbers in tables will vary due to measurement error.***

1. Fill in the table below for the figure you constructed using 15° angles. Use the ruler to measure the length of the AY_n segment.

15° n	measure of angle Y_nAX_n	measure of segment AY_n	measure of segment AX_n	measure of segment X_nY_n
1	90°	$\frac{1}{4}$ in.	1 in.	$1\frac{1}{16}$ in.
2	90°	$\frac{9}{16}$ in.	2 in.	$2\frac{1}{16}$ in.
3	90°	$\frac{13}{16}$ in.	3 in.	$3\frac{1}{8}$ in.
4	90°	$1\frac{1}{16}$ in.	4 in.	$4\frac{1}{8}$ in.
5	90°	$1\frac{5}{16}$ in.	5 in.	$5\frac{3}{16}$ in.
6	90°	$1\frac{5}{8}$ in.	6 in.	$6\frac{3}{16}$ in.

2. Fill in the table below for the figure you constructed using 30° angles. Use the ruler to measure the length of the AY_n segment.

30° n	measure of angle Y_nAX_n	measure of segment AY_n	measure of segment AX_n	measure of segment X_nY_n
1	90°	$\frac{9}{16}$ in.	1 in.	$1\frac{1}{8}$ in.
2	90°	$1\frac{1}{8}$ in.	2 in.	$2\frac{13}{16}$ in.
3	90°	$1\frac{3}{4}$ in.	3 in.	$3\frac{1}{2}$ in.
4	90°	$2\frac{5}{16}$ in.	4 in.	$4\frac{5}{8}$ in.
5	90°	$2\frac{7}{8}$ in.	5 in.	$5\frac{3}{4}$ in.
6	90°	$3\frac{3}{16}$ in.	6 in.	$6\frac{15}{16}$ in.

3. Fill in the table below for the figure you constructed using 45° angles. Use the ruler to measure the length of the AY_n segment.

45° n	measure of angle Y_nAX_n	measure of segment AY_n	measure of segment AX_n	measure of segment X_nY_n
1	90°	1 in.	1 in.	$1\frac{7}{16}$ in.
2	90°	2 in.	2 in.	$2\frac{13}{16}$ in.
3	90°	3 in.	3 in.	$4\frac{1}{4}$ in.
4	90°	4 in.	4 in.	$5\frac{11}{16}$ in.
5	90°	5 in.	5 in.	$7\frac{1}{16}$ in.
6	90°	6 in.	6 in.	$8\frac{1}{2}$ in.

4. Fill in the table below for the figure you constructed using 60° angles. Use the ruler to measure the length of the AY_n segment.

60° n	measure of angle Y_nAX_n	measure of segment AY_n	measure of segment AX_n	measure of segment X_nY_n
1	90°	$1\frac{3}{4}$ in.	1 in.	2 in.
2	90°	$3\frac{7}{16}$ in.	2 in.	4 in.
3	90°	$5\frac{3}{16}$ in.	3 in.	6 in.
4	90°	$6\frac{15}{16}$ in.	4 in.	8 in.
5	90°	$8\frac{11}{16}$ in.	5 in.	10 in.
6	90°	$10\frac{3}{8}$ in.	6 in.	12 in.

5. Fill in the table below for the figure you constructed using 75° angles. Use the ruler to measure the length of the AY^n segment.

75° n	measure of angle Y_nAX_n	measure of segment AY_n	measure of segment AX_n	measure of segment X_nY_n
1	90°	$3\frac{3}{4}$ in.	1 in.	$3\frac{7}{8}$ in.
2	90°	$7\frac{7}{16}$ in.	2 in.	$7\frac{11}{16}$ in.
3	90°	$11\frac{3}{16}$ in.	3 in.	$11\frac{9}{16}$ in.
4	90°	$14\frac{15}{16}$ in.	4 in.	$15\frac{7}{16}$ in.
5	90°	$18\frac{11}{16}$ in.	5 in.	$19\frac{3}{8}$ in.
6	90°	$22\frac{3}{8}$ in.	6 in.	$23\frac{3}{16}$ in.

6. Use a calculator to calculate the ratios listed in the table below using the information from the figure you constructed using 15° angles. Round all your answers to the nearest thousandth.

15° n	ratio AX_n/X_nY_n	ratio AY_n/X_nY_n	ratio AY_n/AX_n	ratio X_nY_n/AX_n	ratio X_nY_n/AY_n	ratio AX_n/AY_n
1	.941	.235	.250	1.063	4.250	4.000
2	.970	.273	.281	1.031	3.667	3.556
3	.960	.260	.271	1.042	3.761	3.692
4	.970	.258	.266	1.031	3.882	3.765
5	.964	.253	.263	1.038	3.952	3.810
6	.970	.263	.271	1.031	3.808	3.692
Average	.963 cos	.257 sin	.267 tan	1.039 sec	3.887 csc	3.753 cot

7. Use a calculator to calculate the ratios listed in the table below using the information from the figure you constructed using 30° angles. Round all your answers to the nearest thousandth.

30° n	ratio AX_n/X_nY_n	ratio AY_n/X_nY_n	ratio AY_n/AX_n	ratio X_nY_n/AX_n	ratio X_nY_n/AY_n	ratio AX_n/AY_n
1	.889	.500	.563	1.125	2.000	1.778
2	.865	.486	.563	1.156	2.056	1.778
3	.857	.500	.583	1.167	2.000	1.714
4	.865	.500	.578	1.156	2.000	1.730
5	.870	.500	.575	1.150	2.000	1.739
6	.865	.495	.573	1.156	2.018	1.745
Average	.868	.499	.573	1.152	2.012	1.747

8. Use a calculator to calculate the ratios listed in the table below using the information from the figure you constructed using 45° angles. Round all your answers to the nearest thousandth.

45° n	ratio AX_n/X_nY_n	ratio AY_n/X_nY_n	ratio AY_n/AX_n	ratio X_nY_n/AX_n	ratio X_nY_n/AY_n	ratio AX_n/AY_n
1	.696	.696	1	1.438	1.438	1
2	.711	.711	1	1.406	1.406	1
3	.706	.706	1	1.417	1.417	1
4	.703	.703	1	1.422	1.422	1
5	.708	.708	1	1.413	1.413	1
6	.706	.706	1	1.417	1.417	1
Average	.705	.705	1	1.419	1.419	1

9. Use a calculator to calculate the ratios listed in the table below using the information from the figure you constructed using 60° angles. Round all your answers to the nearest thousandth.

60° n	ratio AX_n/X_nY_n	ratio AY_n/X_nY_n	ratio AY_n/AX_n	ratio X_nY_n/AX_n	ratio X_nY_n/AY_n	ratio AX_n/AY_n
1	.5	.875	1.750	2	1.143	.571
2	.5	.859	1.719	2	1.164	.582
3	.5	.865	1.729	2	1.157	.578
4	.5	.867	1.734	2	1.153	.577
5	.5	.869	1.738	2	1.151	.576
6	.5	.865	1.729	2	1.157	.578
Average	.5	.867	1.733	2	1.154	.577

10. Use a calculator to calculate the ratios listed in the table below using the information from the figure you constructed using 75° angles. Round all your answers to the nearest thousandth.

75° n	ratio AX_n/X_nY_n	ratio AY_n/X_nY_n	ratio AY_n/AX_n	ratio X_nY_n/AX_n	ratio X_nY_n/AY_n	ratio AX_n/AY_n
1	.258	.968	3.750	3.875	1.033	.267
2	.260	.967	3.719	3.843	1.034	.269
3	.268	.968	3.729	3.854	1.034	.268
4	.268	.968	3.734	3.859	1.033	.268
5	.268	.965	3.738	3.875	1.037	.268
6	268	.965	3.729	3.865	1.036	.268
Average	.265	.967	3.733	3.862	1.035	.268

11. Use a calculator to fill in the table below, rounding all your answers to the nearest thousandth.

angle	sine	cosine	tangent	cotangent	secant	cosecant
15°	.259	.966	.268	3.732	1.033	3.884
30°	.500	.866	.577	1.732	1.155	2.000
45°	.707	.707	1.000	1.000	1.414	1.414
60°	.866	.500	1.732	.577	2	1.155
75°	.966	.259	3.732	.268	3.864	1.035

CONCLUSIONS:

1. Describe the relationship between the first row of the table in Number 11 above with the table in Number 6.

   AX_n/X_nY_n = COS; AY_n/X_nY_n = SIN; AY_n/AX_n = TAN

   X_nY_n/AX_n = SEC; X_nY_n/AY_n = CSC; AX_n/AY_n = COT

2. Describe the relationship between the second row of the table in Number 11 above with the table in Number 7.

   Same as # 1.

3. Describe the relationship between the third row of the table in Number 11 above with the table in Number 8.

   Same as # 1.

4. Describe the relationship between the fourth row of the table in Number 11 above with the table in Number 9.

   Same as # 1.

5. Describe the relationship between the fifth row of the table in Number 11 above with the table in Number 10.

   Same as # 1.

6. How do the results of your investigation compare with the values on the trigonometric table found in a mathematics text?

   With care, they should be close. Remind students that measurements are only accurate to the nearest $\frac{1}{16}$ inch.

SUGGESTIONS FOR FURTHER STUDY:

- Explore the wrapping function and the relationship of this function to the trigonometric functions.
- Investigate relationships between the cosine and cosecant of the various angles.
- Investigate similarity relationships between triangles in each figure.
- Explore applications of the trigonometric functions, such as sound waves.

TEACHER'S GUIDE
WHERE ARE THE IMAGINARY ROOTS?

GOAL: The student will develop an understanding of graphic representations of complex roots of simple polynomials.

STUDENT OBJECTIVES:

- ✓ To construct graphs showing roots of quadratic functions of the form $y = kx^2 + c$, where x is a complex number but y is a real number.
- ✓ To graph relations of the form $x^2 + y^2 = r^2$, where x is a complex number but y is a real number.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are some skill with graphing equations in rectangular coordinate systems and familiarity with conic sections, complex numbers, and graphic interpretations of roots of a function. This investigation can be done by students individually or in cooperative groups. The graphs on the student page of this activity may be cut-out and used to construct the model. However, reproducing the graphs on rectangular coordinate systems on poster board makes the models more durable. Fishing line can be used to suspend the three dimensional models from the ceiling of your classroom.

VOCABULARY: quadratic equations, real roots, imaginary roots, complex numbers, real numbers, rectangular coordinate systems

SUGGESTED PATH FOR REMEDIATION: Students who have trouble producing the graphs for this activity may need additional instruction on graphing conic sections. Computer software for three-dimensional graphing is now readily and inexpensively available. This technology may be very helpful to those who are having trouble visualizing the graphs in three dimensions.

ADDITIONAL RESOURCES: *Precalculus Mathematics: A Graphing Approach*, a product of the Ohio State University Calculator and Computer Precalculus Project, directed by Bert Waites and Frank Demanea (Addison-Wesley Publishing Company), has a section on "Complex Numbers as Zeros," which provides additional information on the topic covered in this investigation.

WHERE ARE THE IMAGINARY ROOTS?

INTRODUCTION: Complex roots in the form $x = a + bi$ have a real part, a, and an imaginary part, b. By graphing these two parts separately and combining the two graphs, you can have a more complete visual model.

PURPOSES:

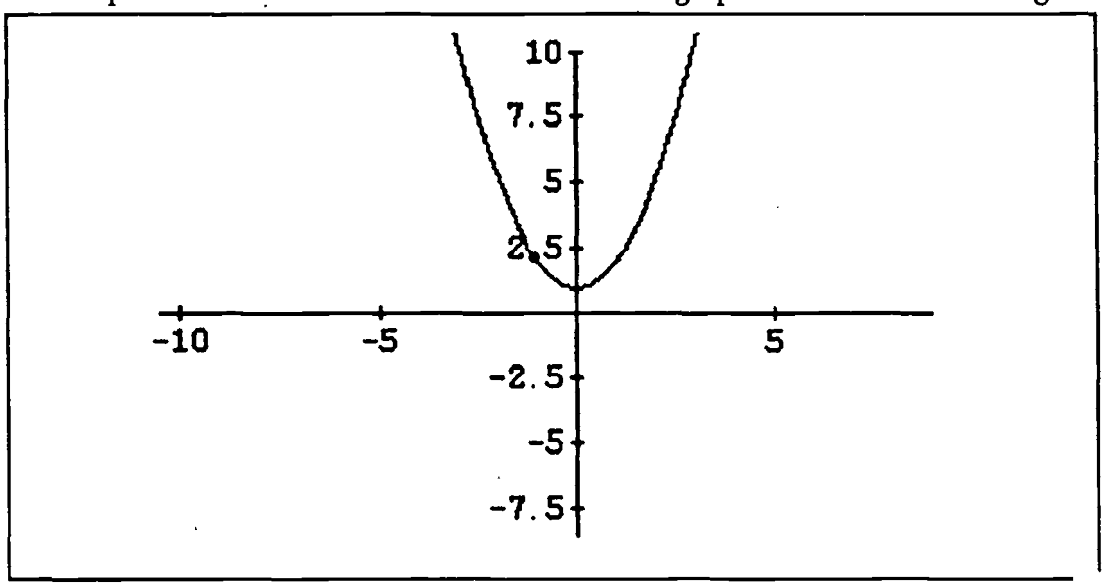
- ✓ Can you construct graphs showing roots of quadratic functions of the form $y = kx^2 + c$, where x is a complex number but y is a real number?
- ✓ Can you graph relations of the form $x^2 + y^2 = r^2$, where x is a complex number but y is a real number?

MATERIALS:

poster board	paste
grid paper	scissors

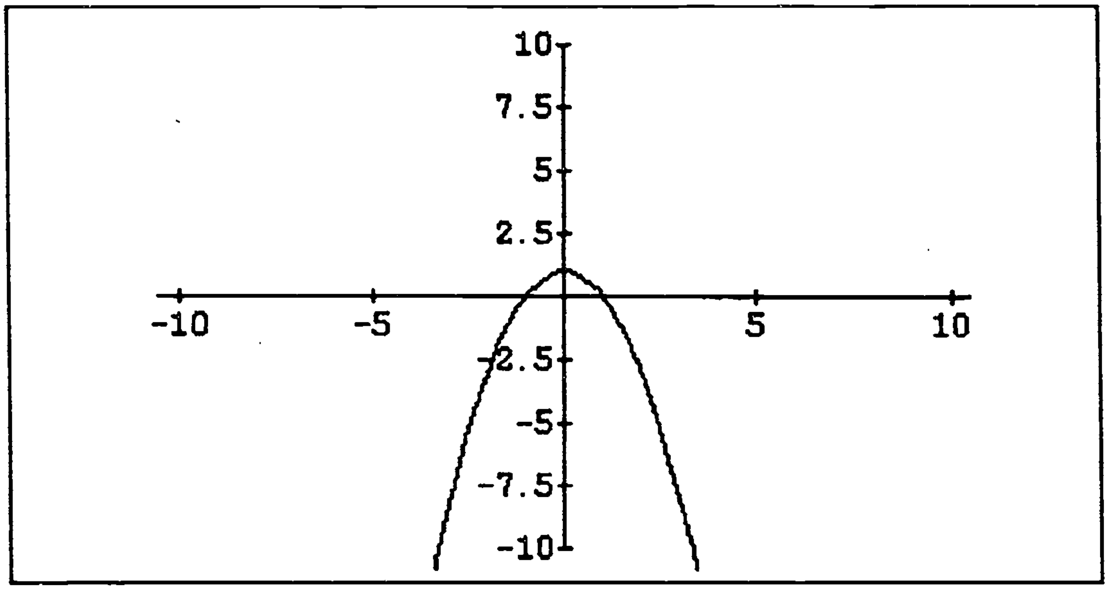
PROCEDURES:

1. Cut-out two six by six-inch squares of poster board and cover them with grid paper on both sides by pasting it on. Draw the horizontal and vertical axes on both sides of each piece of poster board. On both sides of the first piece label the horizontal axis a and the vertical axis y, and on both sides of the other, label the horizontal axis b and the vertical axis y.
2. Using the grid with horizontal axis labeled a, draw the graph of $y = x^2 + 1$ in the real plane where $x = a + bi$ and $b = 0$. Draw this graph on both sides of the grid.



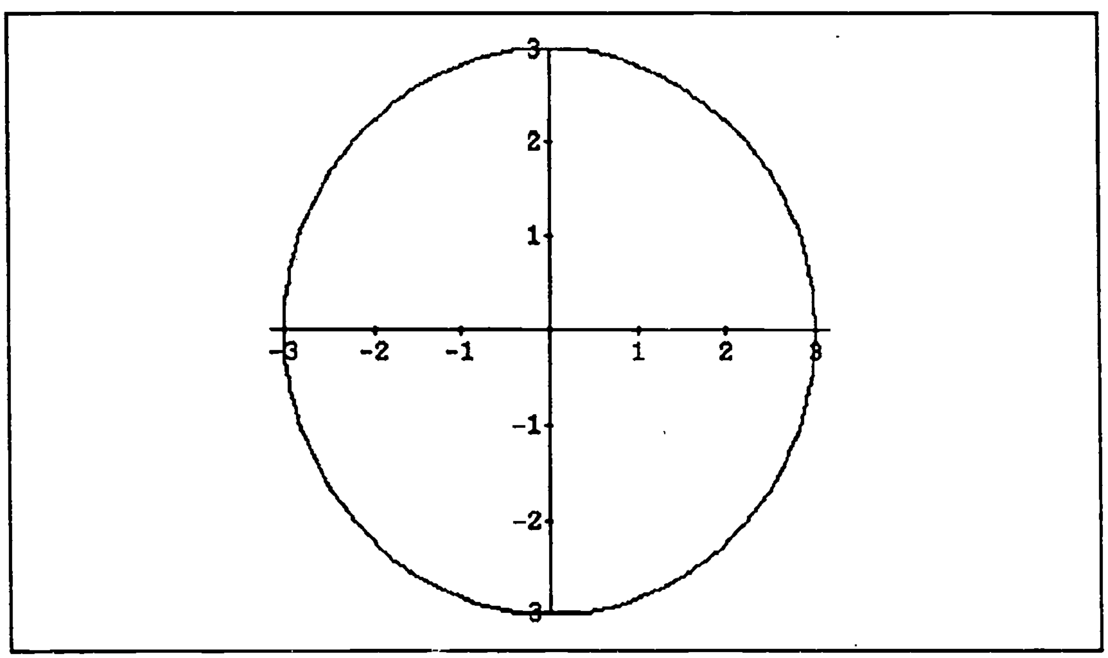
36 - 1

3. Using the grid with horizontal axis labeled b, draw the graph of $y = x^2 + 1$ in the imaginary plane where $x = a + bi$ and $a = 0$. Draw this graph on both sides of the grid.



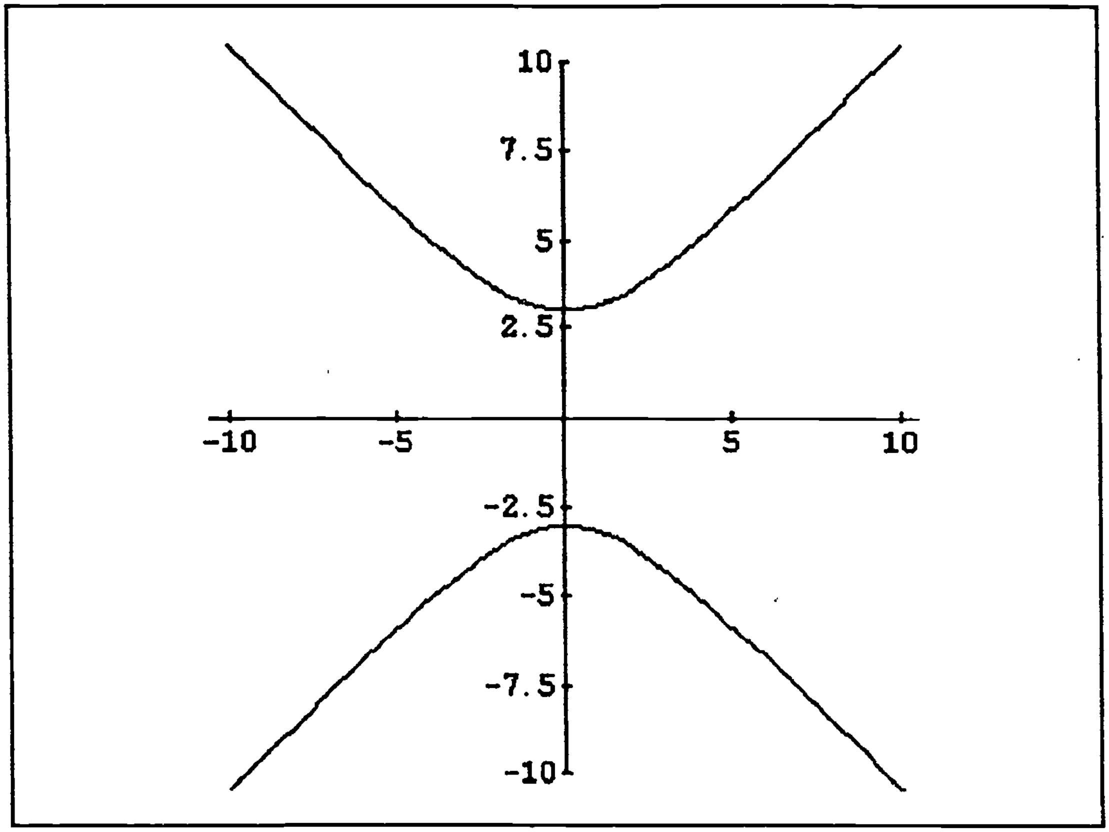
36 - 2

4. Cut the two grids along opposite halves of the y-axis. Cut one from the bottom up to the intesection of the horizontal axis, and the other down to this point. Slide the two grids together at a right angle.



36 - 3

5. Repeat this procedure to obtain a graph of $x^2 + y^2 = 9$ where $x = a + bi$ and y is a real number, that is $(a + bi)^2 + y^2 = 9$. First, let $b = 0$, and graph the circle in the real plane.



36 - 4

6. Then let $a = 0$ and graph the hyperbola in the imaginary plane.
7. Cut the two grids along opposite halves of the y-axis. Slide the two grids together at a right angle.

OBSERVATIONS:

1. How many real roots does $y = x^2 + 1$ have?

   None

2. How many imaginary roots does $y = x^2 + 1$ have?

   Two: $-i, i$

3. How many real roots does $x^2 + y^2 = 9$ have?

   Two: $-3, 3$

4. How many imaginary roots does $x^2 + y^2 = 9$ have?

   Two: $-3i, 3i$

CONCLUSIONS:

1. Describe the graph of $y = x^2 + 1$ in three dimensions: y, a, and b.

   They are cones—one opening up and one opening down.

2. Describe the graph of $x^2 + y^2 = 9$ in three dimensions: y, a, and b.

   It is a sphere with two cones—one on top and one on the bottom.

SUGGESTIONS FOR FURTHER STUDY:

- Graph other equations of the form $y = kx^2 + c$, where x is a complex number but y is a real number.
- Graph other relations of the form $x^2 + y^2 = r^2$, where x is a complex number but y is a real number.
- Graph relations of the form $(x - h)^2 + (y - k)^2 = r^2$, where x is a complex number but y is a real number.
- What would happen if y in your equations were allowed to take on complex values as well as real?

REASONING

TEACHER'S GUIDE
AM I SPEEDING?

GOAL: The student will develop an understanding of linear graphs and the slope-intercept form of linear equations.

STUDENT OBJECTIVES:

- ✓ To collect data showing the relationship between distance and time, when the velocity is, for all practical purposes, constant.
- ✓ To produce a graphic representation of these data.
- ✓ To produce a linear equation in slope-intercept form to fit the data reasonably well.
- ✓ To use this linear equation to predict distance, given time.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are an ability to graph lines in the Cartesian plane and to understand the slope-intercept form of the equation of a line. You may wish to collect the data required, or you may explain to students how data are to be collected and ask that they collect the data during a trip. If neither of these options are available, the data below was collected as prescribed in this activity.

Time (min.)	Odometer Reading miles
0	18.4
4	22.6
7	26.1
11	30.8
14	34.1
17	37.4
19	39.5
23	43.6
25	45.8
28	47.7

A more uniform velocity is possible on a freeway or interstate highway. The odometer reading is used instead of the distance traveled so that the line will have a y-intercept other than zero. This facilitates a discussion of the meaning of y-intercept.

VOCABULARY: graph, x-axis, y-axis, slope, y-intercept, abscissa, ordinate, velocity, distance, time, odometer, approximate

SUGGESTED PATH FOR REMEDIATION: If students have trouble finding the slope of the line fitting the data, ask them to consider the speed they were traveling when the data were collected. Car speed is usually measured in miles per hour, but the time unit is minutes. If they can convert the miles per hour speed to miles per minute, this will be a good starting approximation for the slope of the line. Since slope is defined as the difference between the ordinates divided by the difference between the abscissas, the slope of the line is the change in distance divided by the change in time. This, of course, is the definition of velocity.

ADDITIONAL RESOURCES: *Data Analysis and Statistics Across the Curriculum: Addenda Series, Grades 9-12* (National Council of Teachers of Mathematics) is an excellent source of activities related to data collection and interpretation.

AM I SPEEDING?

INTRODUCTION: Many things in everyday life are linearly related. This activity attempts to demonstrate such a relationship between time and distance traveled when you are going at a relatively constant rate of speed.

PURPOSES:

- ✓ Can data collected during freeway driving be represented by a linear graph?
- ✓ What role does the velocity or speed of travel have on this graph?
- ✓ Can this graph be used to make predictions?

MATERIALS:

car with an odometer
a digital clock
graph paper

PROCEDURES:

1. This activity requires data collected while driving on the freeway. Your teacher may have collected these data for you. The odometer reading (the last four digits will suffice) needs to be recorded as the car enters the freeway. The time for this reading will be recorded as $x = 0$. Every few minutes record the number of minutes which has passed since entering the freeway and the odometer reading. A table is provided on the next page for recording these data. Twelve or more data points are needed. Please make note of any instances during the investigation requiring a significant change in speed.
2. Construct a graph of these data points with time on the x-axis and the odometer reading (distance) on the y-axis.

OBSERVATIONS:

1. Record the data from the freeway driving in the table below.

Time (min.)	Odometer Reading

2. While you were collecting the data, what speed was the car being driven?

____

3. Describe the pattern of the points on the graph.

A line, if speed was reasonably uniform.

____

____

CONCLUSIONS:

1. Can you write a linear equation ($y = mx + b$) that does a reasonably good job of approximating the graph you constructed? The value of b is the initial odometer reading. You may use trial and error to find the slope of this line. You may be surprised to find what it turns out to be!

y = (speed) x + b (initial odometer reading)

____

2. So, what was it?

   The speed.

3. If there are data points that do not fit this line very well, what are some possible explanations?

   The speed of the car may have varied due to traffic conditions.

4. Use your equation to predict the odometer reading if you had driven for an hour.

   Answers will vary.

   Two hours.

   Three hours.

SUGGESTIONS FOR FURTHER STUDY:

- Use linear regression to find the regression equation for the data you collected. Compare this with the equation you wrote in this investigation.
- Study the relationship between velocity and acceleration.

TEACHER'S GUIDE
NUMBERS: Perfect, Deficient, Abundant, And Amicable

GOAL: The student will develop an understanding of the relationship between a number and its divisors.

STUDENT OBJECTIVES:

- ✓ To find as many perfect numbers as practical.
- ✓ To find as many pairs of amicable numbers as practical.

GUIDE TO THE INVESTIGATIONS: A prerequisite skill for this investigation is the ability to find all the divisors of a given integer. A knowledge of prime and composite numbers is helpful. Students may work in groups or individually. If you choose, prizes may be awarded to the student or cooperative group finding the largest perfect number and to the student or group finding the greatest number of pairs of amicable numbers.

VOCABULARY: perfect, deficient, abundant, amicable, divisor, proper divisor, prime, composite number

SUGGESTED PATH FOR REMEDIATION: Students having trouble with the first phase of this investigation are probably having difficulty listing all the proper divisors of numbers. Reviewing the definition of divisor and the procedure for finding divisors in pairs may be helpful. Perhaps a review of divisibility rules may help.

Many students do not realize that all pairs of factors of a given number will have been found by the time they reach the number's square root. For example, all pairs of factors of 75 will have been found by the time you reach 8 because the square root of 75 is less than 9. Some students are reluctant to start the first part of this investigation because they believe that listing all the factors of the numbers will take too long, but it will actually not take very long at all. Therefore, it may help if you show them an example.

ADDITIONAL RESOURCES: *Factors and Multiples*, from the Middle Grades Mathematics Project, provides good number theory activities. These materials were written at the Department of Mathematics, Michigan State University and are distributed by Addison-Wesley Publishing Company.

NUMBERS
Perfect, Deficient, Abundant, And Amicable

INTRODUCTION: The proper divisors of a number are the divisors other than the number itself. Thus, the proper divisors of 6 are 1, 2, and 3, but not 6. If the sum of the proper divisors of a number is equal to the number, then it is called a **perfect number**. If this sum is less than the number, then it is called a **deficient number**. If the sum of the proper factors is greater than the number, the number is called **abundant**. Two numbers are said to be **amicable** if each one is the sum of the proper divisors of the other. From Greek to Medieval times, amicable numbers were believed to have mystical relationships to human friendship.

PURPOSES:

- ✓ What are the perfect numbers less than 100?
- ✓ What are the deficient numbers less than 100?
- ✓ What are the abundant numbers less than 100?
- ✓ Are there any pairs of amicable numbers both of which are less than 100?
- ✓ How many perfect numbers are there?
- ✓ Are there any odd perfect numbers?
- ✓ What is the largest perfect number you can find?
- ✓ How many pairs of amicable numbers can you find?

MATERIALS:

calculator
paper and pencil

PROCEDURES:

1. Make a table similar to that on the following page for the natural numbers from one to one hundred.

Number	Proper Divisors	Sum of Proper Divisors	Type of Number
1	none	none	none
2	1	1	deficient
3	1	1	deficient
4	1, 2	3	deficient
5	1	1	deficient
6	1, 2, 3	6	perfect
7	1	1	deficient
8	1, 2, 4	7	deficient
9	1, 3	4	deficient
10	1, 2, 5	8	deficient
11	1	1	deficient
12	1, 2, 3, 4, 6	16	abundant
13	1	1	deficient

2. Euclid proved that if $2^n - 1$ is a prime number then $N = (2^{n-1})(2^n - 1)$ is perfect and that every even perfect number is of the form $(2^{n-1})(2^n - 1)$. Complete the chart below to find some prime numbers of the form $2^n - 1$, then use these numbers to find some even perfect numbers.

n	$2^n - 1$	Prime ?	$(2^{n-1})(2^n - 1)$
1	1	no	1
2	3	yes	6
3	7	yes	28
4	15	no	120
5	31	yes	496
6	63	no	2 016
7	127	yes	8 128
8	255	no	32 640
9	511	no	130 816
10	1 023	no	523 776

3. A sixteen year old boy was the first person to discover the amicable pair 1 184 and 1 210. Prove that these numbers are amicable by writing each of the sums of the other's proper divisors.

1 184 – 1, 2, 4, 8, 16, 32, 37, 74, 148, 296, 592 Sum 1 210

1 210 – 1, 2, 5, 10, 11, 22, 55, 110, 121, 242, 605 Sum 1 184

OBSERVATIONS:

1. How many perfect numbers are there less than 100? What are they?

   6 and 28

2. How many deficient numbers are there less than 100? What are they?

   2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33, 34, 35

   37, 38, 39, 41, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59, 61, 62, 63, 64, 65, 67,

   68, 69, 71, 73, 74, 75, 76, 77, 79, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 97, 98, 99

3. How many abundant numbers are there less than 100? What are they?

   There are 22:

   12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100

4. How many odd perfect numbers are there less than 100? What are they?

   None.

5. List all the pairs of amicable numbers both of which are less than 100.

   None.

   (220 and 284 are amicable.)

6. Check the numbers you computed in Number 2 of the Procedures section of this investigation. Are they all perfect?

   No.

   6 — 1, 2, 3 — 6

   28 — 1, 2, 4, 7, 14 — 28

   496 — 1, 2, 4, 8, 16, 31, 62, 124, 248 — 496

   8128 — 1, 2, 4, 8, 16, 32, 64, 127, 254, 508, 1016, 2032, 4064 — 8128

CONCLUSIONS:

1. Do you believe that there is a largest perfect number? Why or why not?

   No. Answers vary.

2. Write a paper and/or construct a poster displaying what you have learned about perfect, deficient, abundant, and amicable numbers.

SUGGESTIONS FOR FURTHER STUDY:

- Write a computer program to determine whether a number is perfect, deficient, or abundant. You will first have to create a procedure for finding all the proper divisors of the number. Then the sum of the proper divisors will have to be calculated and compared with the number. If the sum is less than the number, then the computer should print *deficient*, and so forth.
- Do some research and see what other amicable pairs have been discovered. Maybe you can discover a new pair!

TEACHER'S GUIDE
GEOMETRY WITHOUT WORDS

GOAL: The student will develop an understanding of intuitive proof.

STUDENT OBJECTIVES:

- ✓ To construct a model for the theorem: The sum of the measures of the angles of a triangle is 180°.
- ✓ To construct a model for the theorem: The angle bisectors of the three angles of a triangle are concurrent.

GUIDE TO THE INVESTIGATIONS: Students can work individually or in cooperative groups to conduct this investigation. One possible way to manage this activity is to have each student in the class draw a different triangle, assigning certain types of triangles to each cooperative group such as acute, obtuse, right, isosceles, scalene, equilateral, and combinations thereof.

The purpose of this investigation is to develop an intuitive feel for geometric concepts. Intuition is often an important precursor of the ability to develop formal geometric proofs.

VOCABULARY: angle, degree, triangle, vertices, adjacent, straight angle, acute, obtuse, right, isosceles, scalene, equilateral, bisector, concurrent, line, median

SUGGESTED PATH FOR REMEDIATION: This activity is designed to motivate development of formal proof, so the lack of ability to write formal geometric proofs is not essential here. The lack of an adequate geometric vocabulary may prevent students from successfully completing the activity, so it may be necessary for some students to review vocabulary. If students are working cooperatively, it may be possible to do this within the cooperative group. Some students may require greater effort.

ADDITIONAL RESOURCES: *Geometry from Multiple Perspectives: Addenda Series, Grades 9-12*, by Arthur F. Coxford, Jr., Linda Burks, Claudia Giamati, and Joyce Jonik (National Council of Teachers of Mathematics) is a excellent source of activities for the geometry classroom.

GEOMETRY WITHOUT WORDS

INTRODUCTION: A proof is a logical argument from premises (givens) to conclusions. Sometimes a picture or physical model can be used to assist in proving a theorem. There are times when the model may be used without words to do so.

PURPOSES:

✓ Can a model be created to prove the theorem: The sum of the measures of the angles of a triangle is 180 degrees?

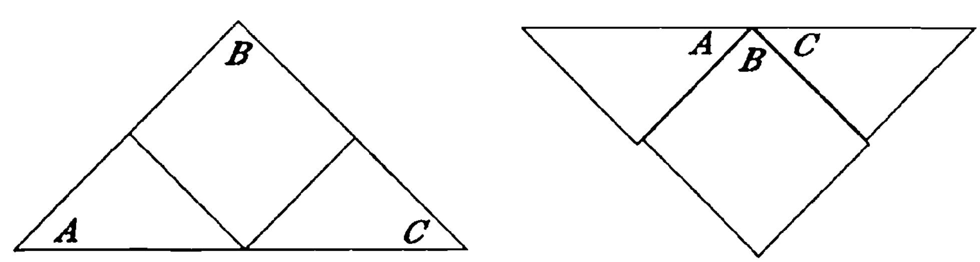
✓ Can a model be created to prove the theorem: The angle bisectors of the three angles of a triangle are concurrent?

MATERIALS:

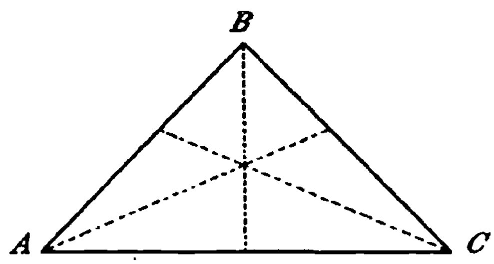
construction paper
wax paper
straight edge
markers
scissors

PROCEDURES:

1. Use a straight edge to draw a triangle on a piece of construction paper. Cut out this triangle and label the vertices inside the triangle A, B, and C. Cut the triangle into 3 pieces in such a way that vertices A, B, and C are each on a separate piece. Rearrange these pieces so that angle A is adjacent to angle B, and angle B is adjacent to angle C. Repeat this process several times using different kinds of triangles. Answer Question 1 in the Observations section and Question 1 in the Conclusions section.



2. Use the straight edge and a felt-tip marker to draw a triangle on a piece of wax paper. Label the vertices of the triangle *A*, *B*, and *C*. Fold the wax paper so that line *AC* is concurrent with line *AB*. Carefully crease the paper along this fold. This crease is the bisector of angle *A*. Repeat this process to produce the bisectors of angle *B* and angle *C*. Mount the wax paper on a piece of construction paper. Repeat this process with various types of triangles. Answer Question 2 in the Observations section and Question 2 in the Conclusions section.



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OBSERVATIONS:

1. In Procedure 1 of this investigation, when angle *A*, angle *B*, and angle *C* are placed adjacent to each other, what kind of angle is formed? What is the measure of the combined angle?

   180° = π radius

2. In Procedure 2 of this investigation, are the bisectors of angle *A*, angle *B*, and angle *C* concurrent? That is, do they all three intersect at a common point?

   Yes

   If so, is this point of concurrence in the interior of the triangle, the exterior of the triangle, or on the triangle?

   Interior

   Was the triangle you started with an acute, obtuse, or right triangle?

   Answers will vary.

CONCLUSIONS:

1. What is the sum of the measures of the three angles of the triangle with which you did Procedure 1 of this investigation?

   180°

   If you had started with a different triangle what do you believe would have been the result of this investigation?

   The sum would still be 180°

2. Are the angle bisectors of the triangle with which you did Procedure two of this investigation concurrent?

   Yes

   If you had started with a different triangle, do you believe that the angle bisector would have been concurrent?

   Yes

SUGGESTIONS FOR FURTHER STUDY:

- Construct a proof without words for the theorem: The medians of triangles are concurrent.
- Develop proofs without words for other theorems.

TEACHER'S GUIDE
HOW LONG IS $\sqrt{2}$?

GOAL: To develop students' understanding of irrational numbers and their positions on the real number line.

STUDENT OBJECTIVE

✓ To develop a procedure for constructing a line segment whose length is of the form $\sqrt{n}$, where n is a positive integer.

GUIDE TO THE INVESTIGATIONS: The prerequisite skills for this activity are the ability to construct a line segment and a perpendicular to a line at a point on the line, the Pythagorean Theorem, the Mean Proportion Theorem, and knowledge of the fact that any angle inscribed in a semicircle is a right angle.

There exists a one-to-one correspondence between points on the number line and real numbers. Since irrational numbers are real numbers, for each irrational number there is a corresponding point on the number line, and a line segment can be constructed which has length represented by that irrational number. In this activity students will construct segments with irrational length $\sqrt{n}$, where n is a positive integer. (Of course, not all numbers of the form $\sqrt{n}$ are irrational: $\sqrt{4} = 2$, for example.)

Students may work alone or in cooperative groups to perform this activity. Each will need several sheets of butcher paper, a straight edge and a compass. Each student or cooperative group should follow the instructions in the Procedures section to construct a segment of length using the radical spiral and then a segment of this length using the Mean Proportion Theorem. Finally, each will construct a number line. Since each started with a line segment of different length, each student or group may have a different unit length, so the number lines will almost certainly be of different proportions. Display these number lines and radical spirals in your classroom or on the bulletin boards in the school halls. One nice result of this activity is that students should realize that the answer to the question "How long is the square root of two?" depends on how long one is. This provides a great opportunity to talk about different measurement systems. If students start with a unit of 1 inch, then they can find the length of $\sqrt{2}$ inches. If they started with a unit of 1 foot, then they can find the length of $\sqrt{2}$ feet, and so forth.

VOCABULARY: natural number, integer, rational and irrational numbers, square root, compass, radical spiral, proportion

SUGGESTED PATH FOR REMEDIATION: The teacher may need to demonstrate the first two or three iterations in the construction of the radical spiral. If this is done

on the overhead or the chalkboard, it will provide a refresher for all students. If cooperative groups are used, a good remediation strategy is to have a student who does not recall the construction procedure to do the actual construction while another member of the group gives explicit instructions. Since the procedure calls for fifteen iterations of the radical spiral, there will be ample opportunities for all group members to perform the construction.

ADDITIONAL RESOURCES: In the *Middle Grades Mathematics Project, Factors and Multiples* (Addison-Wesley, 1986), there is an activity on Factor Pairs (pp. 35–46) which leads students to locate square roots of integers by finding the point of intersection of the line $y = x$ and the curve $xy = n$ where n is a positive integer. This could be used as an alternative approach for finding the length of $\sqrt{n}$.

HOW LONG IS $\sqrt{2}$?

INTRODUCTION: Do you know hov long the square root of two is? One thing you will learn in this activity is that the answer to this question is, "It depends on how long one is."

PURPOSE:

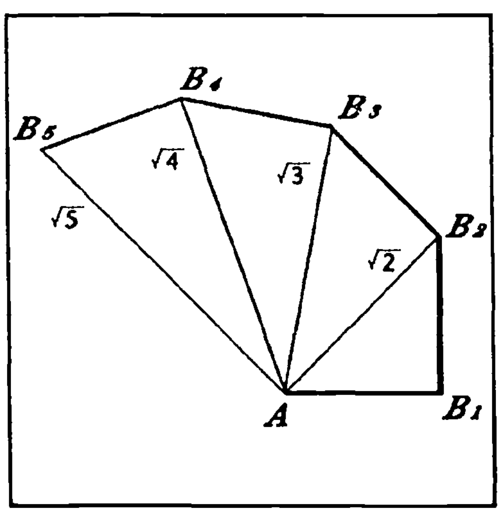
✓ How can you locate irrational numbers such as $\sqrt{2}$ on the number line?

MATERIALS:

butcher paper
compass
straight edge

PROCEDURES:

1. Use the straight edge to draw a line segment. Do not make it too large, because you will let the length of this line segment be one unit for the remainder of this activity, and you are going to need sufficient room to draw longer segments. Neither make it too small, for, if you do, then the drawing which follows will be tedious and hard to see.
2. Construct a copy of the original unit line segment. Label the end points of this segment A and B_1.
3. At the point B_1 construct a perpendicular to the line segment AB_1.
4. Mark off one unit length on this perpendicular beginning at B_1. Label the other endpoint B_2.
5. Use your straight edge to connect A and B_2. What is the length of the line segment AB_2? (Use the Pythagorean Theorem.)
6. At B_2 construct a perpendicular to the line segment AB_2. Mark off one unit on this perpendicular with B_2 as one endpoint, and label the other B_3. Use the straight edge to connect A and B_3. Use the Pythagorean Theorem to calculate the length of AB_3.
7. Repeat this procedure until you have constructed AB_{15}. How long is this line segment? Complete the table on the following page.



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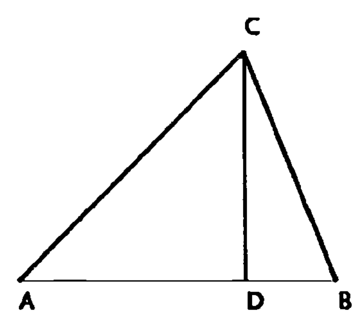
Triangle	Side Length	Side Length	Hypotenuse Length
AB_1B_2	$AB_1 = 1$	$B_1B_2 = 1$	$AB_2 = \sqrt{2}$
AB_2B_3	$AB_2 = \sqrt{2}$	$B_2B_3 = 1$	$AB_3 = \sqrt{3}$
AB_3B_4	$AB_3 = \sqrt{3}$	$B_3B_4 = 1$	$AB_4 = \sqrt{4} = 2$
AB_4B_5	$AB_4 = 2$	$B_4B_5 = 1$	$AB_5 = \sqrt{5}$
AB_5B_6	$AB_5 = \sqrt{5}$	$B_5B_6 = 1$	$AB_6 = \sqrt{6}$
AB_6B_7	$AB_6 = \sqrt{6}$	$B_6B_7 = 1$	$AB_7 = \sqrt{7}$
AB_7B_8	$AB_7 = \sqrt{7}$	$B_7B_8 = 1$	$AB_8 = \sqrt{8} = 2\sqrt{4}$
AB_8B_9	$AB_8 = 2\sqrt{4}$	$B_8B_9 = 1$	$AB_9 = \sqrt{9} = 3$
AB_9B_{10}	$AB_9 = 3$	$B_9B_{10} = 1$	$AB_{10} = \sqrt{10}$
$AB_{10}B_{11}$	$AB_{10} = \sqrt{10}$	$B_{10}B_{11} = 1$	$AB_{11} = \sqrt{11}$
$AB_{11}B_{12}$	$AB_{11}\ \sqrt{11}$	$B_{11}B_{12} = 1$	$AB_{12} = \sqrt{12} = 2\sqrt{3}$
$AB_{12}B_{13}$	$AB_{12} = 2\sqrt{3}$	$B_{12}B_{13} = 1$	$AB_{13} = \sqrt{13}$
$AB_{13}B_{14}$	$AB_{13} = \sqrt{13}$	$B_{13}B_{14} = 1$	$AB_{14} = \sqrt{14}$
$AB_{14}B_{15}$	$AB_{15} = \sqrt{14}$	$B_{14}B_{15} = 1$	$AB_{15} = \sqrt{15}$

8. If n represents a natural number, by repeating the process described above n times, you can produce a radical spiral with the longest segment having length $\sqrt{n}$.

9. Is there a more expedient way to construct a line segment with length $\sqrt{15}$? The Mean Proportion Theorem says that if ABC is a right triangle with right angle at C and CD is perpendicular to AB, then the length of CD is the mean proportion for the lengths of AD and DB.



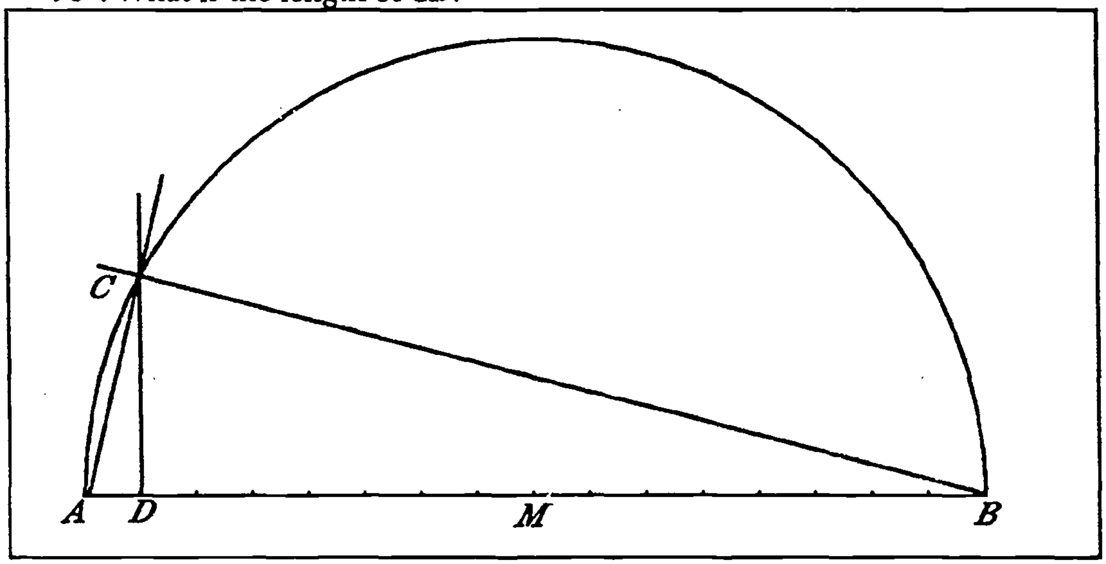
40 - 2

In terms of illustration 40–2, $\frac{\text{the length of } AD}{\text{the length f } CD} = \frac{\text{the length of } CD}{\text{the length of } DB}$. Using this, you can develop a faster method for constructing line segments with length $\sqrt{n}$.

10. Use your straight edge to draw a line on which you copy the one unit line segment sixteen times. Label the left endpoint of the 16-unit long line segment *A* and the other endpoint *B*. Label the point that is one unit from *A* on *AB* as *D*. Label the point that is eight units from A with the letter M. M is the midpoint of *AB*.

11. Place the point of your compass on *M* and set the radius of the compass equal to the length of *AM*. Construct semicircle *AB*.

12. Construct a perpendicular to *AB* at point *D*. Label the intersection of the semicircle and perpendicular as point *C*. Use your straight edge to draw *AC* and *BC*. Angle *ACB* is a right angle, because any angle inscribed in a semicircle is 90°. What is the length of *CD*?



40 - 3

13. You now know two methods for constructing line segments of length $\sqrt{n}$. You may now copy each of these to a number line placing the left end-point of each copy at the point *0* on the number line. The right endpoints will now correspond to the locations of $\sqrt{n}$ for n = 1, ..., 15.

OBSERVATIONS:

1. What is the measure of Angle B_1AB_2?

   45°

2. As each new right triangle in the radical spiral is constructed, what happens to the measure of the angle at vertex *A* of the triangle?

   It decreases.

3. As each new right triangle in the radical spiral is constructed, what happens to the measure of the hypotenuse of the right triangle?

   It increases.

4. Which method is quicker?

The mean proportion theorem.

5. On the number line you constructed, what irrational number did you find between 0 and 1?

None

Between 1 and 2?

$\sqrt{2}, \sqrt{3}$

Between 2 and 3?

$\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$

Between 3 and 4?

$\sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$

CONCLUSIONS:

1. In order to construct a radical spiral with the longest hypotenuse measuring $\sqrt{21}$ units, how many iterations would have to be done? How many for $\sqrt{n}$?

Twenty; $n-1$

2. In the radical spiral, which triangles have a hypotenuse with rational length?

The third, the eighth, and all others whose number can be written as $n^2 - 1$

3. If the Mean Proportion Theorem was used to construct a line segment with measure $\sqrt{21}$ units, how many units long would your initial line need to be? How long for $\sqrt{n}$?

$22, n+1$

4. Write the sequence in which each term represents the length of the hypotenuse of the corresponding right triangle in the radical spiral.

$\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots, \sqrt{n+1}, \ldots$

5. Write an expression for the n-th term of this sequence.

$\sqrt{n+1}$

6. Can all numbers of the form $\sqrt{n}$, where n is a positive integer, be found on the number line?

Yes.

7. Are all numbers of the form $\sqrt{n}$, where n is a positive integer, irrational?

No. If n is a perfect square, $\sqrt{n}$ is rational.

8. How many irrational numbers are there of this form? How many rational? Can you explain that?!

Infinitely many of both. It is not rational to say that $\infty + \infty$ = more than ∞.

9. What do you think of this equation? $\infty + \infty = \infty$

Establish a 1 – 1 correspondence between the following:

natural numbers and odd positive integers

natural numbers and even positive integers

SUGGESTIONS FOR FURTHER STUDY:

- Give a name to the unit length which you used. It is your unit, and you may name it whatever you like. To convert from your unit to centimeters you will need to know how many centimeters are in your unit. Use a calculator to multiply the number of centimeters in your unit times $\sqrt{2}$. Measure on your number line from zero to $\sqrt{2}$ in centimeters. Does this measurement match your calculation? Check for other irrational numbers on your number line. Compare your unit to someone else's unit length. Try to develop a method to convert from your unit length to theirs.
- Use the Mean Proportion Theorem to construct a line segment that has length $3 + 2\sqrt{3}$ units.
- Use a protractor to measure the angles at vertex A in the radical spiral. Use a calculator to compute the tangent of each of these angles. Use a calculator to compute the ratio of measures of the opposite side to the adjacent side of each of the angles in the triangles in the radical spiral. How do the tangents compare to these ratios?

Alpha Hands-On Activities Books Listed According To Grade Levels

ELEMENTARY SCHOOL ACTIVITY BOOKS

- *Hands-On Biology Activities*
- *Hands-On Chemistry Activities*
- *Hands-On Physics Activities*
- *Creative General Science Activities*
- *Reading To Think, Level A-F*
- *P.A.S.S. Program*
- *Hands-On Social Studies Activities*
- *Hands-On Environmental Science Activities*
- *Strategies For Learning: Teaching Thinking Skills Across the Curriculum Through Science*
- *Teaching With Inquiry*
- *Learning Social Studies Through Discrepant Event Inquiry*
- *Celebratiing the Holidays With Science*

MIDDLE SCHOOLS ACTIVITY BOOKS

- *Creative General Science Activities*
- *Hands-On Physical Science Activities*
- *Human Biology: A General Course for Every Student, Vol. 1 & 2*
- *Hands-On Life Science Activities*
- *Hands-On Environmental Science Activities*
- *Earth Science Research Activities*
- *Strategies For Learning: Teaching Thinking Skills Across the Curriculum Through Science*
- *Teaching With Inquiry*
- *Learning Social Studies Through Discrepant Event Inquiry*

HIGH SCHOOL ACTIVITY BOOKS

- *Biology Research Activities*
- *Chemistry Research Activities*
- *Earth Science Research Activities*
- *Physics Research Activities*
- *Hands-On Environmental Science Activities*
- *Introduction to Biology for Every Student, Vol 1-2*
- *Strategies For Learning: Teaching Thinking Skills Across the Curriculum Through Science*
- *Teaching With Inquiry*
- *Learning Social Studies Through Discrepant Event Inquiry*